Computers and Mathematics with Applications 64 (2012) 2912–2916

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On simultaneously nilpotent antiring matrices☆

Jituan Zhou[a,*], Linzhang Lu[b]

[a] School of Mathematics and Computational Sciences, Wuyi University, Jiangmen 529020, China

[b] School of Mathematical Science, Xiamen University, Xiamen 361005, China

ARTICLE INFO

Article history:
Received 9 September 2011
Received in revised form 30 April 2012
Accepted 13 May 2012

Keywords:
Antiring
Nilpotent matrix
Simultaneous nilpotence
Digraph

ABSTRACT

Antirings are an important type of semirings, which generalize Boolean algebra, fuzzy algebra, distributive lattice and incline. In this paper, we study the issue of nilpotent antiring matrices, provide some properties and characterizations of the simultaneous nilpotence for a finite number of antiring matrices, present some methods for calculating the simultaneously nilpotent index of a finite number of antiring matrices. Partial results obtained in this paper generalize and develop the corresponding ones on nilpotent antiring matrices and on simultaneously nilpotent fuzzy matrices (lattice matrices).

1. Introduction

Antirings are an important type of semirings, which were studied in [1] under the name of zerosumfree semirings. Antirings are abundant; for examples, see [2] etc. A number of works on matrices over semirings were published (see e.g. [3–8] etc.). In 1999, Golan [1] described semirings and matrices over semirings comprehensively. The techniques of matrices over semirings are of interest to models of discrete event networks, to optimization theory, and to graph theory; see [9,3] etc.

It is an interesting work to investigate the nilpotent matrices. Many authors have considered this type of matrices over some special antirings (see e.g. [10–19,2,20] etc.). Give'on [11] showed that an $n \times n$ lattice matrix A is nilpotent iff $A^n = 0$. Hashimoto [13] obtained some properties of reduction of nilpotent fuzzy matrices. Tan [19] considered further the reduction of nilpotent matrices over a dually Brouwerian lattice, and generalized some results of Hashimoto [13]. Li [14] and Ren et al. [17] obtained some necessary and sufficient conditions of nilpotent fuzzy matrices, and these results were generalized to nilpotent lattice matrices by Tan [18,19]. Lur et al. [16] showed that a nilpotent fuzzy matrix has acyclic fuzzy digraph representation and provided some properties of nilpotent fuzzy matrices by means of eigenvalues. Also Lur et al. [15,16] characterized the simultaneous nilpotence for a finite number of fuzzy matrices (lattice matrices). Recently Tan [20] characterized the nilpotent matrices over an antiring and the simultaneous nilpotence for a family of matrices over a special antiring in terms of principal submatrices, principal minors, and main diagonals.

In this paper, we continue to consider the simultaneous nilpotence for a finite number of antiring matrices. In Section 3, we present some properties and characterizations for the simultaneous nilpotence and give a method for calculating the simultaneously nilpotent index. In Section 4, we only present some methods on how to calculate the simultaneously nilpotent index.

☆ The paper has been evaluated according to old Aims and Scope of the journal.

* Corresponding author.
E-mail addresses: jtzhou66@126.com, mathzjt@126.com (J. Zhou).

0898-1221/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2012.05.005

2. Preliminaries

In this section, we introduce some definitions and notations.

Definition 1. A nonempty set S with two binary operations $+$ and. is called a semiring if it satisfies the following conditions:

1. $(S, +)$ is an abelian monoid with identity element 0,
2. $(S, .)$ is a monoid with identity element 1,
3. $x(y+z) = xy + xz,\ (y+z)x = yx + zx$ for all $x, y, z \in L$,
4. $0r = r0 = 0$ for all $r \in L$.

A semiring S is called an antiring if $a + b = 0$ implies that $a = b = 0$ for all $a, b \in S$. S is called commutative if $ab = ba$ for all $a, b \in S$; S is called entire if $ab = 0$ implies that either $a = 0$ *or* $b = 0$ for all $a, b \in S$ (see [2]).

The Boolean algebra $(\{0, 1\}, \vee, \wedge)$ is a commutative antiring. The fuzzy algebra $([0, 1], \vee, \wedge)$ is also a commutative antiring. They are all entire. Every distributive lattice is a commutative antiring. The set R^+ of all nonnegative real numbers with the usual operations of addition and multiplication is a commutative antiring which is entire.

Throughout this paper, S always denotes any given commutative antiring with the additive identity 0 and the multiplicative identity 1.

Definition 2. An element $a \in S$ is said to be nilpotent if $a \neq 0$ whereas $a^k = 0$ for some positive integer k. The least positive integer k satisfying $a^k = 0$ is called the nilpotent index of a and denoted by $h(a)$.

If S is entire, then S has no nilpotent elements.

For any positive integer n, $\langle n\rangle$ always stands for the set $\{1, 2, \ldots, n\}$. N denotes the set of all positive integers. Θ stands for the empty set.

The set of all $m \times n$ matrices over S is denoted by $M_{m\times n}(S)$. Especially, we put $M_n(S) := M_{n\times n}(S)$. Let $A = (a_{ik}) \in M_{m\times n}(S)$ and $B = (b_{kj}) \in M_{n\times l}(S)$. The product $A \otimes B \in M_{m\times l}(S)$ is defined by

$$A \otimes B := \left(\sum_{k\in\langle n\rangle} a_{ik}b_{kj}\right).$$

Let $A = (a_{ij}) \in M_{m\times n}(S)$. Denote by a_{ij} or $[A]_{ij}$ the (i, j)-entry of A, and A_{i*} the i-th row of A. Denote by O the zero matrix of suitable order over S. For $A, B \in M_{m\times n}(S)$, $A + B$ is defined by $A + B = (a_{ij} + b_{ij})$ for all $i \in \langle m\rangle, j \in \langle n\rangle$.

For $A \in M_n(S)$. We define the powers of A as follows:

$$A^0 = I_n, \qquad A^l = A^{l-1}A, \quad l = 1, 2, \ldots,$$

where I_n is the identity matrix of order n.

An $n \times n$ antiring matrix is called nilpotent if $A^p = O$ for some $p \in N$ [2]. It is also known that an antiring matrix is nilpotent if and only if $A^n = O$ (if S has no nilpotent elements) [2]. The least positive integer p such that $A^p = O$ is called the nilpotent index of A and denoted by $h(A)$.

In the following definition, we shall extend the nilpotence to a finite number of antiring matrices.

Let $\Re = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset M_n(S)$. For $k \in N$, let $\Re^k$ be the set of all products of matrices in $\Re$ with length k, that is,

$$\Re^k =: \{A_1 \otimes A_2 \otimes \cdots \otimes A_k : A_i \in \Re \ \text{ for } i = 1, 2, \ldots, k\}.$$

Definition 3. Let $\Re = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset M_n(S)$. $\Re$ is said to be simultaneously nilpotent if $\Re^p = \{O\}$ for some $p \in N$. The least integer p such that $\Re^p = \{O\}$ is called the simultaneously nilpotent index of $\Re$ and denoted by $h(\Re)$.

Remark 2.1. (1) If S has no nilpotent elements, then by Corollary 4.1 in [20] $\Re^p = \{O\}$ if and only if $\Re^n = \{O\}$. (2) When $\Re = \{A\}$, we may denote $h(\Re)$ by $h(A)$.

Let $\Re = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset M_n(S)$. The directed graph of $\Re$, denoted by $\Gamma(\Re)$, is the directed graph on n nodes $v_1, v_2, \ldots, v_n$ in which (v_i, v_j) is a directed edge in $\Gamma(\Re)$ from v_i to v_j if and only if $a_{ij} \neq 0$ for some $A = [a_{ij}] \in \Re$. A directed path $\gamma(v_i, v_{l_1}, \ldots, v_{l_{k-1}}, v_j)$ of length k in $\Gamma(\Re)$ is a sequence of $k-$ directed edges $(v_i, v_{l_1}), (v_{l_1}, v_{l_2}), \ldots, (v_{l_{k-1}}, v_j)$. A directed path $\gamma(v_i, v_{l_1}, \ldots, v_{l_{k-1}}, v_j)$ in $\Gamma(\Re)$ is called a cycle if $v_i = v_j$. The directed graph $\Gamma(\Re)$ is called acyclic if $\Gamma(\Re)$ has no cycles. If $\Re = \{A\}$, then $\Gamma(\Re)$ may be denoted by $\Gamma(A)$.

3. Simultaneous nilpotence of a finite number of antiring matrices

The notion of simultaneous nilpotence arises from the study of infinite products of a finite number of matrices which converge to the zero matrix. In this section, we shall present some properties and characterizations of the simultaneous nilpotence for a finite number of antiring matrices and give a method for calculating the simultaneously nilpotent index.