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# Solving Volterra integral equations of the second kind by wavelet-Galerkin scheme[☆]

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ABSTRACT

In this paper, we apply the wavelet-Galerkin method to obtain approximate solutions to linear Volterra integral equations (VIEs) of the second kind. Daubechies wavelets are used to find such approximations. In this approach, we introduce some new connection coefficients and discuss their properties and propose algorithms to evaluate them. These coefficients can be computed just once and applied for solving every linear VIE of the second kind. Convergence and error analysis are discussed and numerical examples illustrate the efficiency of the method.



## 1. Introduction

In recent years, wavelets have played a crucial role in approximating the solution of a wide range of problems arising in science and engineering. Wavelets have been used in numerous areas of applied mathematics as diverse as signal analysis, statistics, computer aided geometric design, image processing and numerical analysis. Glowinski in [1] used wavelets to approximate the solution of a partial differential equation. Wavelet bases are also used for solving integral equations, in which Fredholm equations are investigated more than other types; for e.g. see [2–5]. Wavelets are mostly suited for approximating linear problems, but some numerical results were obtained for nonlinear Fredholm and Volterra equations in [6,7]. The approximate solution of two-dimensional Fredholm equations is discussed in [8].

In a wavelet-Galerkin scheme, wavelet bases are applied with the well-known Galerkin method, in place of other conventional bases like Legendre or Chebyshev bases. This method has been used for approximating PDE problems in [9–12]. The solution of integral equations by the wavelet-Galerkin method is studied by various authors such as Fang in [13], Liang in [14] and Xiao in [15]. Integro-differential equations are also considered in [16].

As we are aware, the wavelet-Galerkin scheme has not been applied for solving VIEs yet. The main difficulty in applying this procedure happens in the evaluation of the connection coefficients which arise in this method. It is difficult and unstable to compute connection coefficients by the numerical evaluation of integrals. Therefore in this paper, we propose algorithms for the exact evaluation of these coefficients.

This paper is organized as follows. In Section 2, some properties of Daubechies wavelets are reviewed. In Section 3, the wavelet-Galerkin scheme is proposed to approximate the solution of a linear VIE of the second kind. The evaluation of connection coefficients is the subject of Section 4. In Section 5, we present the error analysis of this method. The efficiency of this method is shown by providing some numerical experiments in Section 6 and a brief conclusion is presented in Section 7.

☆ The paper has been evaluated according to old Aims and Scope of the journal.

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## 2. Daubechies wavelets

Daubechies wavelets have gained considerable attention in the numerical analysis of partial differential and integral equations because of possessing some useful properties, such as orthogonality, compact support and ability to represent functions at different levels of resolution. In this section we review some properties of Daubechies wavelets.

**Definition 2.1.** A refinable function is a function $\phi : R \to C$ which satisfies a two-scale refinement equation of the form

$$\phi(x) = \sum_k a_k \phi(2x - k). \tag{2.1}$$

The $a_k \in C$ are called scaling or recursion coefficients. The exact values of some of these coefficients are evaluated in [17]. Since $\phi(\cdot)$ has compact support, the series in (2.1) reduces to a finite series as:

$$\phi(x) = \sum_{k=0}^{2g-1} a_k \phi(2x - k),$$

where $g$ denotes the genus of the scaling function.

The refinable function $\phi$ is called orthogonal if

$$\langle \phi(x), \phi(x-k) \rangle = \int_R \phi(x)\phi(x-k)dx = \delta_{0,k}, \quad k \in Z, \tag{2.2}$$

where $\delta_{0,k}$ is the Kronecker delta function. In order to obtain approximations of a function $f(x) \in L^2(R)$, one can use the translated dilations of the scaling function, defined as

$$\phi_{n,l}(x) = 2^{n/2}\phi(2^n x - l), \quad n, l \in Z. \tag{2.3}$$

The set of orthogonal functions $\{\phi_{n,l}(x)\}_{l \in Z}$ for a particular $n$, generates a space $V_n \subset L^2(R)$. Let $P_n$ denote the orthogonal projection $L^2(R) \to V_n$. The vector spaces $V_n (n \in Z)$ have the following properties defining a multiresolution analysis:

1. $V_n \subset L^2(R)$ and $V_n \subset V_{n+1}$
2. $\|f(x) - P_n f(x)\| = \min \|f(x) - g(x)\|$, where $g(x) \in V_n$.
3. $v(x) \in V_n \Leftrightarrow v(2x) \in V_{n+1}$.
4. The projection $P_n f(x)$ converges to $f(x)$ as $n$ tends to infinity:

$$\lim_{n\to\infty} P_n f(x) = f(x) \quad \text{or} \quad \bigcup_{n=0}^{\infty} V_n \text{ is dense in } L^2(R).$$

**Definition 2.2.** The $k$th discrete and continuous moments of $\phi$ are respectively defined by

$$m_k = \frac{1}{2}\sum_l l^k a_l, \tag{2.4}$$

$$M_k = \int x^k \phi(x)dx. \tag{2.5}$$

In this paper by integral sign $\int$, we mean $\int_R$.

**Theorem 2.1** ([18]). *The discrete and continuous moments are related by*

$$M_k = 2^{-k}\sum_{l=0}^{k} \binom{k}{l} m_{k-l} M_l, \tag{2.6}$$

*and we let*

$$\sum_k \phi(x-k) = \int \phi(x)dx = M_0 = 1.$$

## 3. Problem approximation

In this section we propose the wavelet-Galerkin method to approximate the solution of a linear VIE of the second kind.

Let $\phi(x)$ be the Daubechies scaling function of genus $N$ and the support is $[0, L-1]$, where $L = 2N$. Therefore the function $\phi_{n,i}(x) = 2^{-n/2}\phi(2^n x - i)$ has the support $[2^{-n}i, 2^{-n}(i + L - 1)]$. Consider the following VIE of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad 0 \le t \le x \le 1, \tag{3.1}$$