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Analytical solution of the linear fractional system of commensurate order

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ABSTRACT

A useful representation of fractional order systems is the state space representation. For the linear fractional systems of commensurate order, the state space representation is defined as for regular integer state space representation with the state vector differentiated to a real order. This paper presents a solution of the linear fractional order systems of commensurate order in the state space. The solution is obtained using a technique based on functions of square matrices and the Cayley–Hamilton theorem. The technique developed for linear systems of integer order is extended to derive analytical solutions of linear fractional systems of commensurate order. The basic ideas and the derived formulations of the technique are presented. Both, homogeneous and inhomogeneous cases with usual input functions are solved. The solution is calculated in the form of a linear combination of suitable fundamental functions. The presented results are illustrated by analyzing some examples to demonstrate the effectiveness of the presented analytical approach.

1. Introduction

The subject of fractional calculus has gained considerable importance and popularity during the past decades mainly due to its numerous applications in various fields of applied science and engineering. Various materials and processes have been found to be properly described using fractional calculus that provides an excellent instrument for the description of their properties. Fractional derivative concepts have been generally used to model these physical systems, leading to the formulation of fractional differential equations [1–10]. With their growing range of applications, it is important to establish a clear system theory for the fractional order systems, so they may be accessible to the general engineering community. So far, there have been several fundamental works on fractional differential equations that provide a good understanding of the fractional calculus such as the existence, the uniqueness and some methods for solving fractional differential equations [11–14]. More recently, many extensive works have been done on fractional differential equations including the development of efficient analytical and numerical methods and techniques to solve them. The purpose of the analytical methods is to obtain an explicit expression for the general solution of the fractional differential equations [15–24]. The purpose of the numerical methods is the development of a robust and stable numerical scheme for the solution of the fractional differential equation. In recent years, a great deal of effort has been expended in attempting to find accurate and efficient numerical method for their solution that leads to a variety of techniques given for instance in [25–33].

The class of linear fractional dynamic systems is described by general linear fractional differential equations. These types of linear fractional differential equations have been the focus of many studies [15,16,19,21,22,24]. In spite of the finding of their solution was much involved during the last decades; exact solutions cannot be found. Thus approximation and

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numerical techniques have been used extensively. Also, there have been some attempts to create a formal framework for their study, but without the desired generality, coherence and usefulness of the final results.

In this paper, we present an analytical solution of the state-space linear fractional systems of commensurate order given as

$$D^m x(t) = Ax(t) + Be(t), \tag{1}$$

where $D^m x(t)$ is the Caputo fractional derivative of order m such that $0 < m < 1$, $x(t)$ is the n-state vector, A is the $(n \times n)$ state matrix and $e(t)$ is the input.

The objective of this paper is to treat the linear fractional order systems as it is done with usual systems in order to establish a linear fractional order system theory. The solutions of the homogeneous and non-homogeneous cases are obtained using a technique based on functions of square matrices and the Cayley–Hamilton theorem. The general solution is expressed as the linear combination of fundamental functions which are generalized-exponential functions and a kind of functions that play the same role as the classical exponential function. The basic ideas and the derived formulations of the technique are presented. The presented results are illustrated by analyzing some examples to demonstrate the effectiveness of the presented analytical approach. In this work the eigenvalues of the state-space matrix A are all real simples and/or multiples.

2. Fundamental functions

In this context, the fundamental functions will be used in the solution of the state-space linear fractional order systems of Eq. (1). Let λ and m be two real numbers such that $\lambda < 0$ and $0 < m < 1$. For $t \geq 0$, we will call $\mathrm{gexp}_n(t, \lambda, m)$ the generalized exponential function of order n ($n = 1, 2, \ldots$), its Laplace transform is such that

$$L\{\mathrm{gexp}_n(t, \lambda, m)\} = G_n(s) = \frac{1}{(s^m - \lambda)^n}. \tag{2}$$

In order to study the dynamical behavior of the solutions of the state-space linear fractional order systems of Eq. (1), the irrational functions $G_n(s)$ of (2) have to be approximated by rational ones. In [15], the approximation, in a frequency band of interest $[0, \omega_H]$, of the irrational function $G_1(s)$ by a rational function is presented in full details. Hence, from [15], we can write that

$$G_1(s) = \frac{1}{(s^m - \lambda)} = \sum_{i=1}^{2N-1} \frac{k_i}{(s + p_i)}, \tag{3}$$

where the poles p_i ($i = 1, 2, \ldots, 2N - 1$), the residues k_i ($i = 1, 2, \ldots, 2N - 1$) and the number N of the approximation are given by

$$p_i = \frac{(\beta)^{(i-N)}}{\tau_0} \quad \text{and} \quad k_i = \left(\frac{-p_i}{2\pi\lambda}\right)\left[\frac{\sin[(1-m)\pi]}{\cosh\left[m \log\left(\frac{1}{\tau_0 p_i}\right)\right] - \cos[(1-m)\pi]}\right], \tag{4}$$

$$N = \mathrm{Integer}\left[\frac{\log(\tau_0 \omega_{\max})}{\log(\beta)}\right] + 1, \tag{5}$$

with $\tau_0 = (\frac{-1}{\lambda})^{(\frac{1}{m})}$, $\omega_{\max} = 1000\omega_H$ is an approximation frequency and β is the ratio of a pole to a previous one such that $\beta > 1$. By taking the inverse Laplace Transform of Eq. (3), the function $\mathrm{gexp}_1(t, \lambda, m)$ is then obtained as

$$\mathrm{gexp}_1(t, \lambda, m) = L^{-1}\left\{\frac{1}{(s^m - \lambda)}\right\} = L^{-1}\left\{\sum_{i=1}^{2N-1} \frac{k_i}{(s + p_i)}\right\} = \sum_{i=1}^{2N-1} k_i \exp(-p_i t). \tag{6}$$

Once the rational function approximation of the fundamental function $G_1(s)$ is obtained, all the rational function approximations of the fundamental functions $G_n(s)$ ($n = 2, 3, \ldots$) can then be easily derived as

$$G_n(s) = \frac{1}{(s^m - \lambda)^n} = \left[\sum_{i=1}^{2N-1} \frac{k_i}{(s + p_i)}\right]^n = \left[\frac{K_0 \prod_{i=1}^{2N-2} (s + z_i)}{\prod_{i=1}^{2N-1} (s + p_i)}\right]^n = (K_0)^n \left[\frac{\prod_{i=1}^{2N-2} (s + z_i)^n}{\prod_{i=1}^{2N-1} (s + p_i)^n}\right], \tag{7}$$

where the zeros z_i ($i = 1, 2, \ldots, 2N - 2$) and K_0 can be easily calculated by the poles p_i ($i = 1, 2, \ldots, 2N - 1$) and the residues k_i ($i = 1, 2, \ldots, 2N - 1$). By the partial fraction expansions method, we will get

$$G_n(s) = (K_0)^n \left[\frac{\prod_{i=1}^{2N-2} (s + z_i)^n}{\prod_{i=1}^{2N-1} (s + p_i)^n}\right] = \sum_{i=1}^{2N-1} \sum_{j=1}^{n} \frac{k_{ij}}{(s + p_i)^j}, \tag{8}$$