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Meromorphic functions in the unit disc that share slowly growing functions in an angular domain☆

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ABSTRACT

In this paper, we investigate the uniqueness of meromorphic functions in the unit disc that share five distinct meromorphic functions of slow growth in an angular domain.

1. Introduction and main results

Let f be a meromorphic function in $\mathbb{D}_R = \{z : |z| < R\}$, where $0 < R \le \infty$. We adopt the standard notations of Nevanlinna's value distribution theory (see [1] or [2]), such as $T(r, f)$, $N(r, f)$ and $m(r, f)$. Suppose that f and g are two nonconstant meromorphic functions in $\mathbb{D}_R$, α is a meromorphic function in $\mathbb{D}_R$, and $\mathbb{X} \subseteq \mathbb{D}_R$. We say that f and g share α CM (counting multiplicities) in $\mathbb{X}$ provided that $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities in $\mathbb{X}$. Similarly, we say that f and g share α IM (ignoring multiplicities) in $\mathbb{X}$ provided that $f-\alpha$ and $g-\alpha$ have the same zeros in $\mathbb{X}$.

Since Nevanlinna [3] proved the famous five-value theorem and four-value theorem by using his value distribution theory, lots of uniqueness results of meromorphic functions in the complex plane $\mathbb{C}$ have been obtained, which are introduced systematically in [4]. In [5], Zheng firstly took into account the uniqueness dealing with five shared values in some angular domains of $\mathbb{C}$. It is an interesting topic to investigate the uniqueness dealing with shared values in the remaining part of the complex plane removing an unbound closed set (see [6–8], etc.). In [6], Zheng proved the following result by using the Nevanlinna theory in an angular domain.

Theorem A (*See [6]*). *Let f and g be both transcendental meromorphic functions in $\mathbb{C}$. Given one angular domain $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ with $0 \le \alpha < \beta \le 2\pi$ and for some positive number ε and for some $a \in \mathbb{C}_\infty \left(=\mathbb{C}\bigcup\{\infty\}\right)$,*

$$\overline{\lim_{r\to\infty}} \frac{\log n(r, \Omega(\alpha+\varepsilon, \beta-\varepsilon), f = a)}{\log r} > \frac{\pi}{\beta-\alpha},$$

where $n(r, \Omega(\alpha+\varepsilon, \beta-\varepsilon), f = a)$ denotes the number of the roots of $f(z) = a$ in $\{z : |z| < r\} \bigcap \Omega(\alpha+\varepsilon, \beta-\varepsilon)$ counting multiplicities. If f and g share five distinct values a_j $(j = 1, \ldots, 5)$ IM in $\Omega(\alpha, \beta)$, then $f \equiv g$.

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In the sequel, let f be meromorphic in the unit disc $\mathbb{D}_1$ and $\Delta(\theta_0,\delta)$ denote the domain $\{z : |z| < 1\}\bigcap\{z : |\arg z-\theta_0| < \delta\}$, where $0 \le \theta_0 < 2\pi$, $0 < \delta < \pi$. We use $n(r, \Delta(\theta_0,\delta), f = a)$ to denote the number of the roots of $f(z) = a$ in $\Delta(\theta_0,\delta)\bigcap\{z : |z| < r\}$ counting multiplicities.

In [9], we pose the investigation of the uniqueness of meromorphic functions in the unit disc sharing values in an angular domain. By using the conformal mapping and the properties of meromorphic functions in the unit disc, we obtained the following result.

Theorem B. *Let f and g be two meromorphic functions in $\mathbb{D}_1$, $a_j \in \mathbb{C}_\infty$ $(j = 1,\dots,5)$ be five distinct values, and $\Delta(\theta_0,\delta)$ be an angular domain such that for some $a \in \mathbb{C}_\infty$,*

$$\overline{\lim_{r\to 1}}\frac{\log n(r, \Delta(\theta_0,\delta/2), f = a)}{\log\frac{1}{1-r}} > 1.$$

If f and g share a_j $(j = 1,\dots,5)$ IM in $\Delta(\theta_0,\delta)$, then $f \equiv g$.

In this paper, we continue to make the above investigation. Our main purpose is to replace the values a_j $(j = 1,\dots,5)$ in Theorem B by the functions α_j $(j = 1,\dots,5)$ of slow growth with respect to f, and obtain the following theorems. First we introduce some definitions.

In our discussion of the above investigation, the following Ahlfors–Shimizu characteristic is an important tool. Let f be meromorphic in $\mathbb{D}_1$, and $\Delta(\theta,\delta)$ be an angular domain in $\mathbb{D}_1$. Define

$$S_0(r, \Delta(\theta,\delta), f) = \frac{1}{\pi}\int_{\theta-\delta}^{\theta+\delta}\int_0^r\left(\frac{|f'(te^{i\phi})|}{1+|f(te^{i\phi})|^2}\right)^2 tdtd\phi,$$

$$T_0(r, \Delta(\theta,\delta), f) = \int_0^r\frac{S_0(t, \Delta(\theta,\delta), f)}{t}dt.$$

Especially, we define

$$S_0(r,f) = \frac{1}{\pi}\int_0^{2\pi}\int_0^r\left(\frac{|f'(te^{i\phi})|}{1+|f(te^{i\phi})|^2}\right)^2 tdtd\phi,$$

$$T_0(r,f) = \int_0^r\frac{S_0(t,f)}{t}dt.$$

Then from Theorem 1.4 in [1], we have

$$|T(r,f) - T_0(r,f) - \log^+|f(0)|| \le \frac{1}{2}\log 2. \tag{1.1}$$

So by (1.1), we get the following definition.

Definition 1.1. Let f be a meromorphic function in $\mathbb{D}_1$, the order $\sigma(f)$ of f is defined by

$$\sigma(f) = \overline{\lim_{r\to 1}}\frac{\log^+ T(r,f)}{\log\frac{1}{1-r}} = \overline{\lim_{r\to 1}}\frac{\log^+ T_0(r,f)}{\log\frac{1}{1-r}}.$$

Definition 1.2. Let f be a meromorphic function in $\mathbb{D}_1$ of finite order. If for arbitrary small positive number ε, we have

$$\overline{\lim_{r\to 1}}\frac{\log n(r, \Delta(\theta_0,\varepsilon), f = a)}{\log\frac{1}{1-r}} = \sigma(f) + 1$$

for all but at most two exceptional values $a \in \mathbb{C}_\infty$, then $e^{i\theta_0}$ is called a Borel point of order $\sigma(f) + 1$ of f, the ray $L(\theta_0) = \{z : |z| < 1, \arg z = \theta_0\}$ is called a Borel radius of f.

Remark 1.1. In [10], Valiron proved that every meromorphic function of finite order $\sigma(f)$ in the unit disc must have at least one Borel point of order $\sigma(f) + 1$.

Now we give the main results of this paper.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions of finite order in $\mathbb{D}_1$, α_j $(j = 1,\dots,5)$ be five distinct meromorphic functions in $\mathbb{D}_1$ such that $\max_{1\le j\le 5}\{\sigma(\alpha_j)\} < \sigma(f)$, and let $\Delta(\theta_0,\delta)$ $(0 < \delta < \pi)$ be an angular domain containing the Borel radius $L(\theta_0)$ of f. If f and g share α_j $(j = 1,\dots,5)$ IM in $\Delta(\theta_0,\delta)$, then $f \equiv g$.*

If f is a meromorphic function of infinite order in $\mathbb{D}_1$, there exists a precise order $\rho\left(\frac{1}{1-r}\right)$ of f such that (introduced by Hiong, see [11,12])