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On the global Krylov subspace methods for solving general coupled matrix equations

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ABSTRACT

In the present paper, we propose the global full orthogonalization method (Gl-FOM) and global generalized minimum residual (Gl-GMRES) method for solving large and sparse general coupled matrix equations

$$\sum_{j=1}^{p} A_{ij}X_jB_{ij} = C_i, \quad i = 1, \ldots, p,$$

where $A_{ij} \in \mathbb{R}^{m\times m}$, $B_{ij} \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{m\times n}$, $i, j = 1, 2, \ldots, p$, are given matrices and $X_i \in \mathbb{R}^{m\times n}$, $i = 1, 2, \ldots, p$, are the unknown matrices. To do so, first, a new inner product and its corresponding matrix norm are defined. Then, using a linear operator equation and new matrix product, we demonstrate how to employ Gl-FOM and Gl-GMRES algorithms for solving general coupled matrix equations. Finally, some numerical experiments are given to illustrate the validity and applicability of the results obtained in this work.

1. Introduction

In this paper, we consider the general coupled matrix equations of the form

$$\sum_{j=1}^{p} A_{ij}X_jB_{ij} = C_i, \quad i = 1, \ldots, p, \tag{1.1}$$

where $A_{ij} \in \mathbb{R}^{m\times m}$, $B_{ij} \in \mathbb{R}^{n\times n}$, and $C_i \in \mathbb{R}^{m\times n}$, $i, j = 1, 2, \ldots, p$, are large and sparse matrices, $X_i \in \mathbb{R}^{m\times n}$, $i = 1, 2, \ldots, p$, are the unknown matrices. Such problems arise in linear control and filtering theory for continues or discrete-time large-scale dynamical systems. They also play an important role in image restoration and other problems; for more details see [1–5] and the references therein.

Many investigated matrix equations in the literature can be considered as special cases of (1.1). For example, Bouhamidi and Jbilou [1] have considered the generalized Sylvester matrix equation

$$\sum_{j=1}^{p} A_jXB_j = C, \tag{1.2}$$

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and proposed a Krylov subspace method for solving (1.2). In [6], Li and Wang proposed an iterative algorithm for the minimal norm least squares solution to (1.2). Chang and Wang [7] have presented necessary and sufficient conditions for the existence and the expressions for the symmetric solutions of the matrix equations

$$\begin{cases} AX + YA = C, \\ AXA^T + BYB^T = C, \end{cases}$$

and

$$(A^TXA, B^TXB) = (C, D).$$

In [8], Wang et al. have given necessary and sufficient conditions for the existence of constant solutions with bi(skew)symmetric constrains to the matrix equations

$$A_iX - YB_i = C_i, \quad i = 1, 2, \ldots, s,$$

and

$$A_iXB_i - C_iYD_i = E_i, \quad i = 1, 2, \ldots, s.$$

A good survey of the methods to solve special cases of the general coupled matrix (1.1) can be found in [9].

It is easy to see that the general coupled matrix (1.1) is equivalent to

$$\sum_{j=1}^{p} (B_{ij}^T \otimes A_{ij})\mathrm{vec}(X_j) = \mathrm{vec}(C_i), \quad i = 1, \ldots, p, \tag{1.3}$$

where $\otimes$ denotes the Kronecker product operator and $\mathrm{vec}(Z) = (z_1^T, z_2^T, \ldots, z_m^T)^T$ for $Z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^{m\times n}$. Obviously, the coefficient matrix of the linear system (1.3) is of order pmn and can be solved by iterative methods such as the methods based on the Krylov subspace methods like the GMRES [10]. Evidently, the size of the linear system (1.3) would be huge even for moderate values of m, n and p. Therefore, it is more preferable to employ an iterative method for solving the original system (1.1) instead of the linear system (1.3). Note that system (1.1) has a unique solution if and only if the coefficient matrix of the linear system (1.3) is nonsingular. Throughout this paper we assume that the system (1.1) has a unique solution.

In [9], Dehghan and Hajarian have presented an iterative method to solve the general coupled matrix equations (1.1) over generalized bisymmetric matrix group $(X_1, X_2, \ldots, X_p)$. In [11], a gradient based algorithm and a least square based iterative algorithm have been presented for solving (1.2). Ding and Chen [12] used the hierarchical identification principle to construct iterative solutions to the coupled linear matrix equation (1.1). In [13], Zhou et al. proposed an iterative method for finding weighted least squares solutions to system (1.1). A gradient based iterative algorithm for solving coupled matrix equations has been presented by Zhou et al. in [14]. Recently, Zhang in [4] has extended the CGNE [15] and Bi-CGSTAB [15] algorithms to solve (1.1).

In [2], the global Krylov subspace methods have been originally presented for solving a linear system of equations with multiple right-hand sides. It is well-known that the global Krylov subspace methods outperform other iterative methods for solving such systems when the coefficient matrix is large and nonsymmetric. On the other hand, the global Krylov subspace methods are also effective when applied for solving large and sparse linear matrix equations; for more details see [1,16,17] and the references therein. Therefore, we are interested in employing the global Krylov subspaces for solving (1.1) when the coefficient matrices are large and sparse. To do so, we first define the linear operator $\mathcal{M}$ as follows

$$\mathcal{M} : \mathbb{R}^{m\times n} \times \cdots \times \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times pn},$$
$$X = (X_1, X_2, \ldots, X_p) \to \mathcal{M}(X) = (\mathcal{A}_1(X), \mathcal{A}_2(X), \ldots, \mathcal{A}_p(X)),$$

where

$$\mathcal{A}_i(X) = \sum_{j=1}^{p} A_{ij}X_jB_{ij}, \quad i = 1, 2, \ldots, p.$$

Using the linear operator $\mathcal{M}$, we rewrite Eq. (1.1) as

$$\mathcal{M}(X) = C, \tag{1.4}$$

where $C = (C_1, C_2, \ldots, C_p)$. In the next sections, we utilize the linear matrix operator $\mathcal{M}$ to present Gl-FOM and Gl-GMRES algorithms for solving (1.1). More precisely, we focus on the solution of Eq. (1.4) instead of Eq. (1.1).

The rest of the paper is organized as follows. In Section 2, we first recall some necessary definitions and notations, then a new inner product is presented. We also introduce a new matrix product and give some of its properties. Section 3 is devoted to employing the Gl-FOM and Gl-GMRES algorithms for solving Eq. (1.4). In Section 4, some numerical experiments are given to show the efficiency of the proposed algorithms. Finally, the paper finishes with a brief conclusion in Section 5.