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The order of convexity of some general integral operators

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ABSTRACT

The main object of the present paper is to discuss some extensions of certain integral operators and to obtain their order of convexity. Several other closely related results are also considered.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane and denote by $H(U)$ the class of the olomorphic functions in U. Consider $\mathcal{A} = \{f \in H(U) : f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in U\}$ be the class of analytic functions in U and $S = \{f \in \mathcal{A} :\ f \text{ is univalent in } U\}$.

Consider S^* the class of starlike functions in the unit disk, defined by

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U \right\}.$$

Definition 1.1. A function $f \in S$ is a *starlike function of order* α, $0 \le \alpha < 1$ and denote this class by $S^*(\alpha)$ if f verifies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in U).$$

Denote with K the class of convex functions in U, defined by

$$K = \left\{ f \in \mathcal{A} :\ \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0,\ z \in U \right\}.$$

Definition 1.2. A function $f \in S$ is a *convex function of order* α, $0 \le \alpha < 1$ and denote this class by $K(\alpha)$ if f verifies the inequality

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad (z \in U).$$

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It is well known that $K(\alpha) \subset S^*(\alpha) \subset S$.

Recently, Frasin and Jahangiri in [1] defined the family $B(\mu, \alpha),\ \mu \geq 0,\ 0 \leq \alpha < 1$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu} - 1 \right| < 1 - \alpha, \quad (z \in U). \tag{1}$$

The family $B(\mu, \alpha)$ is a comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $B(1, \alpha) \equiv S^*(\alpha)$.

In this paper we will obtain the order of convexity of the following general integral operators:

$$G(z) = \int_0^z \prod_{i=1}^{n} (g_i(u))^{\beta-1}\, \mathrm{d}u, \tag{2}$$

$$G_n(z) = \int_0^z \prod_{i=1}^{n} (g_i(u))^{\beta_i-1}\, \mathrm{d}u, \tag{3}$$

$$H(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{g_i(u)}{u} \right)^{\frac{1}{\beta}} u^{n(\beta-1)}\, \mathrm{d}u, \tag{4}$$

$$F(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{g_i(u)}{u} \right)^{\beta_i} \mathrm{d}u, \tag{5}$$

where the functions $g_1(u),\ g_2(u), \ldots, g_n(u)$ are in $B(\mu, \alpha)$.

In order to prove our main results, we recall the following lemma:

Lemma 1.1 *([2] General Schwarz Lemma).* *Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} \cdot |z|^m \quad (z \in U_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m,$$

where θ is constant.

2. Main results

Theorem 2.1. *Let $g_i(z)$ be in the class $B(\mu, \alpha),\ \mu \geq 1,\ 0 \leq \alpha < 1$ for all $i = 1, 2, \ldots, n$. If $|g_i(z)| \leq M_i$ $(M_i \geq 1,\ z \in U)$, for all $i = 1, 2, \ldots, n$ then the integral operator*

$$G(z) = \int_0^z \prod_{i=1}^{n} (g_i(u))^{\beta-1}\, \mathrm{d}u$$

is in $K(\delta)$, where

$$\delta = 1 - |\beta - 1| \cdot (2 - \alpha) \sum_{i=1}^{n} M_i^{\mu-1} \tag{6}$$

and $|\beta - 1| \cdot (2 - \alpha) \sum_{i=1}^{n} M_i^{\mu-1} < 1,\ \beta \in \mathbb{C}$.

Proof. Let $g_i(z)$ be in the class $B(\mu, \alpha),\ \mu \geq 1,\ 0 \leq \alpha < 1$ for all $i = 1, 2, \ldots, n$. It follows from (2) that

$$\frac{G''(z)}{G'(z)} = (\beta - 1) \sum_{i=1}^{n} \frac{g_i'(z)}{g_i(z)}$$

and, hence

$$\begin{aligned} \left| \frac{zG''(z)}{G'(z)} \right| &\leq |\beta - 1| \left(\sum_{i=1}^{n} \left| \frac{zg_i'(z)}{g_i(z)} \right| \right) \\ &\leq |\beta - 1| \left(\sum_{i=1}^{n} \left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu} \right| \cdot \left| \left(\frac{g_i(z)}{z} \right)^{\mu-1} \right| \right). \end{aligned} \tag{7}$$