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Second order duality for the variational problems under $\rho-(\eta,\theta)$-invexity

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ABSTRACT

In this paper, we introduce the concept of second order duality for the variational problems using $\rho-(\eta,\theta)$-invexity type conditions. Weak, strong and converse duality results of Mangasarian and Mond–Weir type of variational problems are established under $\rho-(\eta,\theta)$-invexity assumptions. Many examples and counterexamples are illustrated to justify our work.

1. Introduction

The study of second order duality is useful due to the computational advantage over first order duality as it gives bounds for the value of the objective function when approximations are used (see [1–3]). Mangasarian [2] formulated a class of second and higher order dual problems of nonlinear programming problems, and established the duality results under inclusion conditions. Hanson [4] defined the second order invexity for differentiable functions and proved the duality results for a pair of mathematical programs. Many researchers [5–8] have discussed various properties, extensions, and applications of generalized invex functions, for example in [9], Zalmai talked about $\rho-(\eta,\theta)$-invexity functions. Chen [10] formulated the second order dual for the class of constrained variational problems and established the duality results (weak, strong, converse) under invexity assumptions.

In this paper we extend the second order duality results (weak, strong, converse) of Chen [10] under generalized $\rho-(\eta,\theta)$-invexity assumptions. We establish second order duality results (weak, strong, converse) of Mangasarian and Mond–Weir type. We also discuss many examples and counterexamples to verify our results.

2. Notation and preliminaries

Let $I=[a,b]$ be an interval (through out this paper). Consider the function $f(t,x(t),\dot{x}(t))$, where $x:I\rightarrow\mathbb{R}^n$ and $\dot{x}$ denotes the derivative of x with respect to t. Here t is an independent variable. All vectors will be taken as column vectors. The symbol z^T stands for the transpose of a vector z. Denote the first partial derivatives of f with respect to $x(t)$, and $\dot{x}(t)$ by

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f_x and $f_{\dot{x}}$, respectively, that is,

$$f_x = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}, \qquad f_{\dot{x}} = \begin{pmatrix} \frac{\partial f}{\partial \dot{x}_1} \\ \frac{\partial f}{\partial \dot{x}_2} \\ \vdots \\ \frac{\partial f}{\partial \dot{x}_n} \end{pmatrix}.$$

Denote f_{xx}, the Hessian matrix of f with respect to $x(t)$ and g_x the $m \times n$ Jacobian matrix with respect to x. Similarly $f_{\dot{x}}$, $f_{\dot{x}x}$, $f_{x\dot{x}}$ and $g_{\dot{x}}$, are also defined. Let $S(I, \mathbb{R}^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \kappa + \int_a^t u(s)\mathrm{d}s, \tag{1}$$

where κ is a given boundary value; thus $\frac{\mathrm{d}}{\mathrm{d}t} = D$ except at discontinuities.

Definition 2.1. The scalar functional $H(x) = \int_a^b h(t, x(t), \dot{x}(t))\mathrm{d}t$ is said to be $\rho - (\eta, \theta)$-invex in x and $\dot{x}$, if there exist $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $\eta = 0$ at $t = a$ and $t = b$, $\theta : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$, such that

$$H(x) - H(u) \geq \int_a^b \Big\{ \eta(t, x(t), u(t))^T h_x(t, u(t), \dot{u}(t)) + \left(\frac{\mathrm{d}}{\mathrm{d}t} \eta(t, x(t), u(t))^T \right) h_{\dot{x}}(t, u(t), \dot{u}(t)) + \rho \|\theta(t, x(t), u(t))\|^2 \Big\} \mathrm{d}t.$$

It follows that every invex function is $\rho - (\eta, \theta)$-invex but the converse is not true, which follows from the following counterexample (Example 2.1).

Example 2.1. Let $f : [0, 1] \times [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ be given by

$$f(t, x(t), \dot{x}(t)) = -x^2(t)t.$$

The function $\int_0^1 f(t, ., .)\mathrm{d}t$ is not invex with respect to any $\eta : [0, 1] \times [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ at $u(t) = 0$. But $\int_0^1 f(t, ., .)\mathrm{d}t$ is a $\rho - (\eta, \theta)$-invex function for $\rho \leq -1$, with respect to $\eta : [0, 1] \times [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ and $\theta : [0, 1] \times [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ given by

$$\eta(t, x(t), u(t)) = x(t) + u(t),$$
$$\theta(t, x(t), u(t)) = \sqrt{t(x(t) + u(t))}.$$

Definition 2.2. The scalar functional $H(x) = \int_a^b h(t, x(t), \dot{x}(t))\mathrm{d}t$ is said to be $\rho - (\eta, \theta)$-pseudo-invex in x and $\dot{x}$, if there exist $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $\eta = 0$ at $t = a$ and $t = b$, $\theta : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$, such that

$$\int_a^b \Big\{ (\eta(t, x(t), u(t)))^T h_x(t, u(t), \dot{u}(t)) + \left(\frac{\mathrm{d}}{\mathrm{d}t} \eta(t, x(t), u(t))^T \right) h_{\dot{x}}(t, u(t), \dot{u}(t)) + \rho \|\theta(t, x(t), u(t))\|^2 \Big\} \mathrm{d}t \geq 0$$
$$\Rightarrow H(x) \geq H(u).$$

It is noted that every $\rho - (\eta, \theta)$-invex function is $\rho - (\eta, \theta)$-pseudo-invex but the converse is not true, which follows from the following counterexample (Example 2.2).

Example 2.2. Let $f : I \times [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ be defined by

$$f(t, x(t), \dot{x}(t)) = -x^3(t) - x(t).$$

Let the functions $\eta : I \times [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ and $\theta : I \times [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ be given by

$$\eta(t, x(t), u(t)) = \begin{cases} 4(x(t) - u(t)), & \text{if } u(t) > x(t) \\ -(x(t) + u(t)), & \text{if } u(t) = x(t) \\ \frac{1}{8}(x(t) - u(t)), & \text{if } u(t) < x(t) \end{cases}$$