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A five-field finite element formulation for nearly inextensible and nearly incompressible finite hyperelasticity

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ABSTRACT

A novel Hu–Washizu type 5-field virtual work principle for nearly inextensible and almost incompressible finite hyperelasticity is developed and implemented. The formulation is implemented in an *hp*-adaptive code providing the proper flexible environment for finite elements with variable order and mixed interpolation. The novel implementation includes residual based error estimation and mesh adaptivity. In the fully constrained limit the formulation provides the constraint manifold setting of hyperelasticity with the simple internal kinematic constraints of inextensibility and incompressibility. A study using a semi-inverse analytical solution and *h*-refinements and *p*-enrichments corroborates the convergence characteristics of the 5-field implementation. A new closed form solution for pure torsion of a circular cylindrical tube with inextensible fibres is derived and used for verification. As typical applications are found in e.g. soft tissue biomechanics, passive pressurisation of an ellipsoidal geometry resembling the left ventricle of a rabbit heart is analysed to demonstrate the capability of modelling transversely isotropic materials.

1. Introduction

Rivlin is one of the main contributors to nonlinear continuum mechanics. His work on the mechanics of rubber (in the 1940s and 50s) established the basis of isotropic incompressible finite hyperelasticity [1]. Spencer, Boehler, Zheng, Zhang and Rychlewski and others extended that work to anisotropic finite hyperelasticity [2, Ch. 8–10], [3,4]. They introduced extended invariant integrity bases describing fibre reinforced materials (in the 1970s and 90s), for a condensed exposition, see Holzapfel [5]. The limiting case of transverse isotropy implies a simple inextensibility. The contemporary theory of simple kinematic constraints is due to Truesdell and Noll [6]. Podio-Guidugli [7] and Carlson and Tortorelli [8] added a geometric–algebraic structure in terms of the constraint manifold. The computational form of Rivlin's work was formulated by Simo, Taylor and Pister [9] in 1985. For a limited compressibility, the unimodular right Cauchy–Green tensor due to Flory [10] plays a central role. It induces traceless (deviatoric) stresses. Supplementing the kinematic set by the volume ratio induces the orthogonal spherical stresses. The concept is generalised herein adding a strong transverse isotropy. The unimodular and simply stretch-less right Cauchy–Green tensor constructed in [11], where the resulting basic five field formulation in the continuous case was presented, is here implemented and employed.

Many soft biological tissues, for example the media and adventitia of arteries, display near-incompressibility and a strong anisotropy due to the presence of collagen fibres [12]. The tensile stiffness of the fibres may become larger than

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the usually assigned constant modulus in dilatation at finite extensions. This situation calls for an auxiliary treatment of the deviatoric fibre-tension and the volume-preserving stretch conjugate pair, in analogy with the mixed formulation for near incompressibility, as already prepared in [11]. There, however, a preliminary $(U, \bar{p})$ two field finite element implementation[1] in terms of the displacement and mean pressure was verified and supported by analytical solutions. In other words, the volume ratio J, the deviatoric fibre tension ϱ and the volume preserving stretch $\tilde{\bar{\lambda}}$ were determined from the displacement. Here the full $(U, \bar{p}, J, \varrho, \tilde{\bar{\lambda}})$ five field mixed formulation including auxiliary interpolations for the deviatoric fibre tension ϱ and the volume preserving fibre stretch $\tilde{\bar{\lambda}}$ is completed, presented and verified. It induces stresses that are traceless and tensionless in the preferred direction. Adding the fibre stretch to the kinematic set induces the deviatoric fibre-tension stress response function. The total fibre tension is determined as the sum of the mean pressure and the deviatoric fibre tension. The novel transversely isotropic and compressible material description that is presented provides the reactive and work-performing stresses of the limiting inextensible and incompressible form derived using the theory for constrained continua [7,8].

Structures including rubber-like materials provided a wide spectrum of real world applications of (nearly) incompressible finite elasticity. Soft tissue biomechanics has in the same spirit provided applications and a request for numerical solutions of boundary value problems including strongly anisotropic nearly incompressible finite elastic materials. The approach has to date predominantly been restricted to use finite element formulations for nearly incompressible rubber-like materials extending the material description to cover actual anisotropies. The approach is today considered standard and available in commercial software like ABAQUS® which provides the necessary library of acknowledged element constructs, for near incompressibility, and suitable materials.

Soft tissue, for example the media- and adventitia-layers in arteries, display near-incompressibility and a strong anisotropy due to the presence of collagen fibres. The popular Holzapfel–Gasser–Ogden (HGO) model [12] available in ABAQUS is often used for soft tissue. The tensile stiffness of the fibres grows exponentially with stretch in the HGO-model and may become larger than the usually assigned constant modulus in dilatation at finite extensions. This situation calls for an auxiliary treatment of the deviatoric fibre tension and the volume-preserving stretch conjugate pair, in analogy with the mixed formulation for near incompressibility. The (low-order) displacement based finite element approach used in engineering codes thus becomes inappropriate approaching strong anisotropy. Extensional locking is a numerical problem similar to volumetric locking but has up to now received much less attention. Computational inextensible finite hyperelasticity is essentially an open field.

In order to enable separate approximations for fibre tension and stretch, a unimodular and simply stretchless right Cauchy–Green tensor was constructed in [11] for the case of strong transverse isotropy, generalising the Simo–Taylor–Pister $\{\boldsymbol{U}, \bar{p}, \tilde{J}\}$ formulation to a five-field formulation including the volume-preserving stretch $\tilde{\bar{\lambda}}$ and the energy conjugate deviatoric fibre tension ϱ. In passing we mention that a somewhat similar idea was recently launched by Al-Kinani, Hartmann and Netz in [13] but revoked shortly afterwards [14] due to inherent problems with the elasticities in that construct.

In [11, Sect. 8] the numerical examples were however performed with a preliminary discretisation, where $\tilde{\bar{\lambda}}$ and ϱ were still displacement based. In this work we present the full mixed FE-implementation five-field formulation. To our knowledge the presented mixed five-field $(U, \bar{p}, J, \varrho, \tilde{\bar{\lambda}})$ FE-implementation is novel. We find support of our formulation in the so-called constraint manifold theory [7,8] for simultaneous incompressibility and inextensibility. Moreover, our formulation provides a mixed FE-formulation in the vicinity of simultaneous incompressibility and inextensibility, extending the STP-formulation [9].

Our five-field Hu–Washizu formulation [11] is here restated in a lighter and more transparent form, as a virtual work principle. It is our basis for a higher-order mixed finite element setting. A standard representation is used for the displacement while the four auxiliary fields are approximated by square integrable test and trial basis functions. A Bubnov–Galerkin formulation is set up. Provided the compressibility and extensibility are finite the four auxiliary unknowns $(\bar{p}, J, \varrho, \tilde{\bar{\lambda}})$ can be eliminated at the element level. The static condensation is performed at virtually no cost. The perturbed mixed method is then of the same type as those in [9,15,16], sometimes called projection penalty methods. Here we extend the concept with higher-order approximations. We extend the mixed displacement pressure higher-order finite element constructs for nearly incompressible finite elasticity introduced by Brink and Stein [17] for finite elastic strong transverse isotropy. In passing it should be mentioned that pure displacement based higher-order finite element approaches have successfully been used in near incompressible finite elasticity avoiding volumetric locking [18] using approximation orders $p \geq 4$, see Düster et al. [19] and Yosibash and Priel [20].

In the fully constrained limit the five-field principle yields the expected double saddle point problem. Plausible element constructs are proposed, implemented and verified.

A semi-inverse analysis is used to corroborate the soundness of the implementation and to show the benefit of higher order approximation. The convergence characteristics obtained support the expectations. A closed form solution for pure torsion of a tube made of a fibre reinforced neo-Hookean Generalised Standard Reinforcing Material (GSRM) is derived and

[1] In FEAP a $(U, \bar{p}, J)$ formulation was employed.