Computers and Mathematics with Applications 72 (2016) 64–75

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On global smooth solution for generalized Zakharov equations ☆

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ARTICLE INFO

Article history:
Received 17 January 2016
Received in revised form 24 March 2016
Accepted 23 April 2016
Available online 19 May 2016

Keywords:
Global solutions
Generalized Zakharov equations
Initial value problem

ABSTRACT

This paper considers the existence and uniqueness of the global smooth solution for the initial value problem of generalized Zakharov equations in dimension two. By means of a priori integral estimates, Galerkin method and compactness theory, one has the existence and uniqueness for small initial data.

1. Introduction

The Zakharov equations, derived by Zakharov in 1972 [1], describe the propagation of Langmuir waves in an unmagnetized plasma. The usual Zakharov system defined in space time $\mathbb{R}^{d+1}$ is given by

$$\begin{aligned} &iE_t + \Delta E = nE, \\ &n_{tt} - \Delta n = \Delta |E|^2, \end{aligned}$$

where $E : \mathbb{R}^{d+1} \to \mathbb{C}^d$ is the slowly varying amplitude of the high-frequency electric field, and $n : \mathbb{R}^{d+1} \to \mathbb{R}$ denotes the fluctuation of the ion-density from its equilibrium.

This system attracted many scientists' wide interest and attention [2–12]. In [7], Larkin and Tronco formulate on a half-strip an initial boundary value problem for the Zakharov–Kuznetsov equation. Existence and uniqueness of a regular solution as well as the exponential decay rate of small solutions as $t \to \infty$ are proven. In [8], the Exp-function method is employed to the Zakharov–Kuznetsov equation as a $(2+1)$-dimensional model for nonlinear Rossby waves. In [9], Ismail Aslan deals with the negative order KdV equation and the generalized Zakharov system and derives some further results using the so-called first integral method. By means of the established first integrals, some exact traveling wave solutions are obtained in a concise manner. In [11], Dem zo, Jahrestag der DDR gewidmet studied the following generalized Zakharov system, and established the global existence for Cauchy problem.

$$\begin{aligned} &i\varepsilon_t + \varepsilon_{xx} + (\alpha - n)\varepsilon = 0, \\ &v_t + \left(\frac{1}{2}v^2 - \beta v_x + n + |\varepsilon|^2\right)_x = 0, \\ &n_t + v_x = 0. \end{aligned}$$

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http://dx.doi.org/10.1016/j.camwa.2016.04.037

In this paper, we are interested in studying the following generalized Zakharov system in dimension two.

$$i\varepsilon_t + \Delta\varepsilon - n\varepsilon = 0, \tag{1}$$

$$v_t + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \mathrm{grad}\varphi(v) - \Delta v + \nabla(n + |\varepsilon|^2) = 0, \tag{2}$$

$$n_t + \nabla \cdot v = 0, \tag{3}$$

with initial data

$$\varepsilon|_{t=0} = \varepsilon_0(x), \qquad v|_{t=0} = v_0(x), \qquad n|_{t=0} = n_0(x), \tag{4}$$

where $\varepsilon(x, t) = (\varepsilon_1(x, t), \varepsilon_2(x, t), \ldots, \varepsilon_N(x, t))$ is an N-dimensional complex valued unknown functional vector, $v(x, t) = (v_1(x, t), v_2(x, t))$ is an 2-dimensional real-valued unknown functional vector, $n(x, t)$ is a real-valued unknown function, $\varphi(s)$ is a real function, $x \in \mathbb{R}^2$.

The obtained results may be useful for better understanding the nonlinear coupling between the quantum ion-acoustic waves and the quantum Langmuir waves in two-dimension space. Now we state the main results of the paper.

Theorem 1. *Suppose that*

(1) $\varepsilon_0(x) \in H^{l+2}(\mathbb{R}^2)$, $v_0(x) \in H^{l+1}(\mathbb{R}^2)$, $n_0(x) \in H^{l+1}(\mathbb{R}^2)$, $l \geq 4$.
(2) $\|\varepsilon_0(x)\|_{L^2}^2 < \sigma\|\psi(x)\|_{L^2}^2$, *where* $0 < \sigma < 1$, $\psi(x)$ *is a solution of the equation* $\Delta\psi - \psi + \psi^3 = 0$.
(3) $|\mathcal{H}(0)|$ *small enough, where*

$$\mathcal{H}(0) = \frac{1}{2}\|v_0\|_{L^2}^2 + \frac{1}{2}\|n_0\|_{L^2}^2 + \|\nabla\varepsilon_0\|_{L^2}^2 + \int_{\mathbb{R}^2} n_0|\varepsilon_0|^2 \mathrm{d}x,$$

(4) $\varphi(v) \in C^{l+2}$, $\varphi(0) = 0$, *and* $\left|\partial_v^\alpha \varphi(v)\right| \leq C(|v| + 1)$, $|\alpha| = 2, \ldots, l + 2$.

Then there exists a unique global smooth solution of the initial value problem (1)–(4),

$$\varepsilon(x, t) \in L^\infty(0, T; H^{l+2}) \cap W^{1,\infty}(0, T; H^l),$$
$$n(x, t) \in L^\infty(0, T; H^{l+1}) \cap W^{1,\infty}(0, T; H^l),$$
$$v(x, t) \in L^\infty(0, T; H^{l+1}) \cap L^2(0, T; H^{l+2}) \cap W^{1,\infty}(0, T; H^{l-1}).$$

For the sake of convenience of the following contexts, we set some notations. For $1 \leq q \leq \infty$, we denote $L^q(\mathbb{R}^d)$ the space of all q times integrable functions in $\mathbb{R}^d$ equipped with norm $\|\cdot\|_{L^q(\mathbb{R}^d)}$ or simply $\|\cdot\|_{L^q}$ and $H^{s,p}(\mathbb{R}^d)$ the Sobolev space with norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^d)}$. If $p = 2$, we write $H^s(\mathbb{R}^d)$ instead of $H^{s,2}(\mathbb{R}^d)$. Let $(f, g) = \int_{\mathbb{R}^n} f(x) \cdot \overline{g(x)} dx$, where $\overline{g(x)}$ denotes the complex conjugate function of $g(x)$.

The paper is organized as follows: In Section 2, we make a priori estimates of the problem (1)–(4). In Section 3, first of all, we obtain the existence and uniqueness of the global generalized solution of the problem (1)–(4) by Galerkin method. Next, the existence and uniqueness of the global smooth solution of the problem are obtained.

2. A priori estimates

In this section, we will derive a priori estimates for the solution of the system (1)–(4).

Lemma 1. *Suppose that* $\varepsilon_0(x) \in L^2(\mathbb{R}^2)$. *Then for the solution of Problem (1)–(4) we have*

$$\|\varepsilon(\cdot, t)\|_{L^2}^2 = \|\varepsilon_0(x)\|_{L^2}^2.$$

Proof. Taking the inner product of (1) and ε, it follows that

$$(i\varepsilon_t + \Delta\varepsilon - n\varepsilon, \varepsilon) = 0 \tag{5}$$

since

$$\mathrm{Im}(i\varepsilon_t, \varepsilon) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\|_{L^2}^2, \qquad \mathrm{Im}(\Delta\varepsilon - n\varepsilon, \varepsilon) = 0$$

hence from (5), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon(\cdot, t)\|_{L^2}^2 = 0,$$

we thus get the lemma. □