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# Improved PPHSS iterative methods for solving nonsingular and singular saddle point problems☆

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ABSTRACT

Based on the parameterized preconditioned Hermitian and skew-Hermitian splitting iteration method (PPHSS), proposed by Li et al. (2014) for saddle point problems, an improvement on the PPHSS method (IPPHSS) is presented in this paper. By adding a block lower triangular matrix to the coefficient matrix on two sides of the first equation of the PPHSS iterative scheme, both the number of iterations and the consume time are decreased. We provide the convergence and semi-convergence analysis of the IPPHSS method, which show that this method is convergence and semi-convergence if the related parameters satisfy suitable restrictions. Furthermore, we discuss the spectral properties of the corresponding preconditioned matrix of the IPPHSS method. Finally, numerical examples show that the IPPHSS method is better than the PPHSS, PHSS and AHSS methods both as a solver and as a preconditioner for the GMRES method.



## 1. Introduction

Consider the following large and sparse saddle point problem

$$\mathcal{A}u = \begin{pmatrix} A & B \\ -B^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{C}^{m\times m}$ is Hermitian positive definite, $B \in \mathbb{C}^{m\times n}$, $p \in \mathbb{C}^m$ and $q \in \mathbb{C}^n$ with $n \le m$. It follows that the saddle point problem (1) is nonsingular when $B$ is of full column rank and singular when $B$ is rank deficient [1].

The saddle point problem (1) is important and arises in a variety of scientific and engineering applications, such as mixed finite element approximation of elliptic partial differential equations, optimal control, computational fluid dynamics, weighted least-squares problems, electronic networks, computer graphics etc.; see [1–3] and references therein.

When $B$ in (1) is full rank, i.e., the saddle point problem (1) is nonsingular, a number of iteration methods and their numerical properties have been discussed to solve the saddle point problem (1) in the literature, such as SOR-like methods [4–7], Uzawa-type methods [4,5,8–11], Hermitian and skew-Hermitian splitting (HSS) methods [12] and its variants [13–17], RPCG iteration methods [18–20] and Krylov subspace iteration methods [21] with high-quality preconditioners [1,2].

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When $B$ in (1) is rank deficient, i.e., the saddle point problem (1) is singular. In recent years, there has been a surge of interest in solving singular saddle point problem (1). In [22–25], the authors applied the Uzawa-type methods to solve singular saddle point problem. Yang et al. [26] discussed the Uzawa-HSS method for singular saddle point problems. Chen and Ma [27] investigated a generalized shift-splitting preconditioner for singular saddle point problems. Wang and Zhang [28] presented a preconditioned AHSS iteration method for singular saddle point problems.

Recently, based on the Hermitian and skew-Hermitian splitting (HSS) of matrix $\mathcal{A}$, that is

$$H = \frac{1}{2}(\mathcal{A} + \mathcal{A}^*) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \frac{1}{2}(\mathcal{A} - \mathcal{A}^*) = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix},$$

some HSS-type methods, such as HSS [12,14], HSS-like [15], PHSS [16], were proposed. In order to improve the convergence rate of the PHSS method, Bai and Golub [17] introduced a two-parameter PHSS method, called accelerated HSS (AHSS) iteration method, to solve the saddle point problem (1). Recently, Wang et al. [29] proposed the IGPHSS method, which improves the convergence rates of the AHSS method.

Considering that the properties of matrices $H$ and $S$ are different, Li et al. [30] considered the parameterized preconditioned Hermitian and skew-Hermitian splitting (PPHSS) method by using two iteration parameters $\alpha$ and $\beta$ instead of one parameter $\alpha$ in the PHSS method, which improves the convergence and extends the possibility to optimize the iterative process, and the PPHSS leads to the PHSS when the two parameters are equal. Subsequently, Chao and Zhang [31] discussed the PPHSS iteration method for singular saddle point problems. In this paper, the PPHSS method is reviewed and an improvement is made, which accelerates the speed of solution.

**The PPHSS method**: Given initial guess $x^{(0)} \in \mathbb{C}^m$ and $y^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$, until the iteration sequence $\{(x^{(k)^*}), (y^{(k)^*})\}$ is convergent, the matrix form of the PPHSS algorithm is:

$$\begin{cases} (\alpha P + H)\begin{pmatrix} x^{\left(k+\frac{1}{2}\right)} \\ y^{\left(k+\frac{1}{2}\right)} \end{pmatrix} = (\alpha P - S)\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + b, \\ (\beta P + S)\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = (\beta P - H)\begin{pmatrix} x^{\left(k+\frac{1}{2}\right)} \\ y^{\left(k+\frac{1}{2}\right)} \end{pmatrix} + b, \end{cases} \tag{2}$$

where $\alpha$ and $\beta$ are given positive constants and matrix $P$ as follows

$$P = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix},$$

with $Q \in \mathbb{C}^{n\times n}$ being an Hermitian positive definite matrix. The iteration matrix is

$$\tilde{L}(\alpha, \beta) = (\beta P + S)^{-1}(\beta P - H)(\alpha P + H)^{-1}(\alpha P - S),$$

and the iteration scheme is:

$$\begin{cases} x^{\left(k+\frac{1}{2}\right)} = \dfrac{\alpha}{1+\alpha}x^{(k)} + \dfrac{1}{\alpha+1}A^{-1}(p - By^{(k)}), \\ y^{\left(k+\frac{1}{2}\right)} = y^{(k)} + \dfrac{1}{\alpha}Q^{-1}(B^*x^{(k)} - q), \\ y^{(k+1)} = \beta D^{-1}Qy^{\left(k+\frac{1}{2}\right)} + D^{-1}\left(\left(1 - \dfrac{1}{\beta}\right)B^*x^{\left(k+\frac{1}{2}\right)} + \dfrac{1}{\beta}B^*A^{-1}p - q\right), \\ x^{(k+1)} = \dfrac{\beta-1}{\beta}x^{\left(k+\frac{1}{2}\right)} + \dfrac{1}{\beta}A^{-1}(p - By^{(k+1)}), \end{cases} \tag{3}$$

where $D = \beta Q + \frac{1}{\beta}B^*A^{-1}B$.

The remainder of this paper is organized as follows. In Section 2, we propose the Improved PPHSS (IPPHSS) method for solving saddle point problem (1). The convergence properties of the IPPHSS method for solving nonsingular saddle point problems and the choice of iteration parameters are discussed in Section 3. We study the spectral property of the corresponding preconditioned matrix in Section 4. The semi-convergence conditions of the IPPHSS method for solving singular saddle point problems will be given in Section 5. In Section 6, numerical examples are provided to examine the feasibility and effectiveness of the IPPHSS method. Finally, some conclusions are drawn.

## 2. The improved PPHSS method

As we all known, the Jacobi and Gauss–Seidel methods are basic iterative methods [32]. Let $G = (g_{ij})$ be a nonsingular $n \times n$ complex matrix with $g_{ii} \neq 0 (i = 1, 2, \ldots, n)$, $d \in \mathbb{C}^n$, thus the solution $x$ of $Gx = d$ exists in $\mathbb{C}^n$ and is unique. We express the matrix $G$ as the form $G = D_1 - E - F$, where $D_1 = \mathrm{diag}(g_{11}, g_{22}, \ldots, g_{nn})$, and $E$ and $F$ are, respectively, strictly