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# Existence and multiplicity of solutions for a class of $(\phi_1, \phi_2)$-Laplacian elliptic system in $\mathbb{R}^N$ via genus theory

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ABSTRACT

In this paper, we investigate the following nonlinear and non-homogeneous elliptic system involving $(\phi_1, \phi_2)$-Laplacian

$$\begin{cases} -\mathrm{div}(\phi_1(|\nabla u|)\nabla u) + V_1(x)\phi_1(|u|)u = F_u(x, u, v) & \text{in } \mathbb{R}^N, \\ -\mathrm{div}(\phi_2(|\nabla v|)\nabla v) + V_2(x)\phi_2(|v|)v = F_v(x, u, v) & \text{in } \mathbb{R}^N, \\ (u, v) \in W^{1,\Phi_1}(\mathbb{R}^N) \times W^{1,\Phi_2}(\mathbb{R}^N) & \text{with } N \geq 2, \end{cases}$$

where the functions $V_i(x)(i = 1, 2)$ are bounded and positive in $\mathbb{R}^N$, the functions $\phi_i(t)t(i = 1, 2)$ are increasing homeomorphisms from $\mathbb{R}^+$ onto $\mathbb{R}^+$, and the function $F$ is of class $C^1(\mathbb{R}^{N+2}, \mathbb{R})$ and has a sub-linear Orlicz–Sobolev growth. By using the least action principle, we obtain that system has at least one nontrivial solution. When $F$ satisfies an additional symmetric condition, by using the genus theory, we obtain that system has infinitely many solutions.



## 1. Introduction and main results

In this paper, we investigate the existence and multiplicity of solutions for the following nonlinear and non-homogeneous elliptic system involving $(\phi_1, \phi_2)$-Laplacian:

$$\begin{cases} -\mathrm{div}(\phi_1(|\nabla u|)\nabla u) + V_1(x)\phi_1(|u|)u = F_u(x, u, v) & \text{in } \mathbb{R}^N, \\ -\mathrm{div}(\phi_2(|\nabla v|)\nabla v) + V_2(x)\phi_2(|v|)v = F_v(x, u, v) & \text{in } \mathbb{R}^N, \\ (u, v) \in W^{1,\Phi_1}(\mathbb{R}^N) \times W^{1,\Phi_2}(\mathbb{R}^N) & \text{with } N \geq 2, \end{cases} \tag{1.1}$$

where $\phi_i$ $(i = 1, 2) : [0, \infty) \to [0, \infty)$ are two functions which satisfy:

$(\phi_1)$ $\phi_i \in C^1[0, \infty)$, $t\phi_i(t) \to \infty$ as $t \to \infty$;
$(\phi_2)$ $t \to \phi_i(t)t$ are strictly increasing;
$(\phi_3)$ $1 < l_i := \inf_{t>0} \frac{t^2\phi_i(t)}{\Phi_i(t)} \leq \sup_{t>0} \frac{t^2\phi_i(t)}{\Phi_i(t)} =: m_i < \min\{N, l_i^*\}$, where $\Phi_i(t) := \int_0^t s\phi_i(s)ds$, $t \in [0, \infty)$ and $l_i^* := \frac{l_i N}{N-l_i}$,
$V_i$ $(i = 1, 2) : \mathbb{R}^N \to \mathbb{R}^+$ are continuous and
$(V)$ there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \leq \min\{V_1(x), V_2(x)\} \leq \max\{V_1(x), V_2(x)\} \leq c_2, \quad \text{for all } x \in \mathbb{R}^N,$$

$F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a $C^1$ function and $F(x, 0, 0) = 0$, for all $x \in \mathbb{R}^N$.

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Set $\phi_2 = \phi_1, v = u, V_2 = V_1$ and $F(x, u, v) = F(x, v, u)$. Let $\Phi_1(t) := \int_0^t s\phi_1(s)ds, t \in [0, \infty)$. Then system (1.1) reduces to the following quasilinear elliptic equation:

$$\begin{cases} -\mathrm{div}(\phi_1(|\nabla u|)\nabla u) + V_1(x)\phi_1(|u|)u = f(x, u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,\Phi_1}(\mathbb{R}^N) & \text{with } N \geq 2. \end{cases} \tag{1.2}$$

Under assumptions $(\phi_1)$–$(\phi_2)$, Eq. (1.2) may be allowed to possess more complicated nonlinear or non-homogeneous operator $\Phi_1$, which can be used to model many phenomena (see [1,2]):

(1) $p$-Laplacian: $\Phi_1(t) = |t|^p$, $1 < p < N$;
(2) $(p, q)$-Laplacian: $\Phi_1(t) = |t|^p + |t|^q$, $1 < p < q < N$;
(3) nonlinear elasticity: $\Phi_1(t) = (1 + t^2)^\gamma - 1, \gamma > \frac{1}{2}$;
(4) plasticity: $\Phi_1(t) = t^\alpha(\log(1 + t))^\beta,\ \alpha \geq 1,\ \beta > 0$;
(5) generalized Newtonian fluids: $\Phi_1(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta ds, 0 \leq \alpha \leq 1, \beta > 0$.

Therefore, Eq. (1.2) or equations like (1.2) have caused great interest among scholars in recent years. When operator $\Phi_1$ is not homogeneous, an Orlicz–Sobolev space setting may be applied for this type of equation (see Section 2). On a bounded domain $\Omega$, we refer the reader to [1,3–12] and the references therein for more information. Especially, in Clément et al. [3], the authors firstly proved the existence of nontrivial solution by variational method when $V_1(x) = 0$ and the nonlinear term $f$ satisfies (AR)-condition. On the whole space $\mathbb{R}^N$, when $V_1(x)$ is bounded, the main difficulty for this type of equation is the lack of compactness of the Sobolev embedding. A usual way to overcome this difficulty is to reconstruct the compactness embedding theorem, which can be done by choosing the radially symmetric function space as the working space (see [2,13,14]).

For the system case, to the best of our knowledge, on the bounded domain $\Omega$, there are only two papers to consider the existence of solutions for systems like (1.1) (see [15,16]). However, on the whole space $\mathbb{R}^N$, there is no paper to study the existence and multiplicity of solutions for systems like (1.1). In this paper, we will study the existence and multiplicity of solutions for system (1.1) under sub-linear Orlicz–Sobolev growth. To overcome the difficulty of lacking compactness of the Sobolev embedding, we will refer to some views in [17]. In [17], Chen and Tang studied the existence and multiplicity of solutions for a class of Kirchhoff-type equation:

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \triangle u + u = f(x, u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) & \text{with } N = 2 \text{ or } 3, \end{cases} \tag{1.3}$$

where constants $a > 0$, $b > 0$, and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. To be precise, they obtained the following results:

**Theorem A** (*See [17, Theorem 1.1]*). *Assume that $f$ satisfies*

$(F_1)^*$ *$f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constants $1 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 2$ and functions $a_i \in L^{\frac{2}{2-\gamma_i}}(\mathbb{R}^N, [0, +\infty))$ $(i = 1, 2, \ldots, m)$ such that*

$$|f(x, u)| \leq \sum_{i=1}^m a_i(x)|u|^{\gamma_i - 1}, \quad \text{for all } (x, u) \in \mathbb{R}^N \times \mathbb{R};$$

$(F_2)^*$ *there exist an open set $J \subset \mathbb{R}^N$ and three constants $\gamma_0 \in (1, 2)$, $\delta > 0$ and $\eta > 0$ such that*

$$F(x, z) \geq \eta|z|^{\gamma_0}, \quad \text{for all}(x, z) \in J \times [-\delta, \delta],$$

*where*

$$F(x, z) := \int_0^z f(x, s)ds, \quad \text{for all } (x, z) \in \mathbb{R}^N \times \mathbb{R}.$$

*Then Eq. (1.3) possesses at least one nontrivial weak solution.*

**Theorem B** (*See [17, Theorem 1.2]*). *Assume that $f$ satisfies $(F_1)^*$ and $(F_2)^*$, then Eq. (1.3) possesses infinitely many nontrivial weak solutions provided that $F$ satisfies*

$(F_3)^*$

$$F(x, -z) = F(x, z), \quad \text{for all}(x, z) \in \mathbb{R}^N \times \mathbb{R}.$$

To overcome the difficulty of lacking compactness, Chen and Tang assumed that the nonlinear term $f$ satisfies a sub-linear growth condition $(F_1)^*$ so that they did not work in the radially symmetric function space. Motivated by such idea in [17], we assume that the nonlinear term $F$ in system (1.1) satisfies a corresponding sub-linear growth condition in Orlicz–Sobolev space (called sub-linear Orlicz–Sobolev growth for short) so that we do not work in the radially symmetric function space,