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Local well-posedness for Boussinesq approximation with shear dependent viscosities in 3D

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ABSTRACT

In this paper, we prove the existence and uniqueness of regular weak solutions for a nonlinear Boussinesq system in a small time interval. The viscosity is assumed to be with a p-structure depending on the shear rate. We mainly work on 3D case with periodic boundary and initial value conditions. The $W^{2,p}$-type regularity for velocity and H^2-type regularity for temperature were obtained for $\frac{7}{5} < p \leq 2$.

1. Introduction

1.1. Formulation of the problem

The flow of a viscous, heat-conducting under the force of gravity is governed by a system of balance equations for mass, momentum and internal energy. There have been numerous attempts to provide a suitable approximation for these systems and few approximation in fluid mechanics have proved as useful and successful in predicting observed phenomena as the Boussinesq approximation. In the so-called Boussinesq equation, the system is reduced to the Navier–Stokes equation for homogeneous, incompressible fluid coupled to a semilinear heat equations (see [1,2]) and it has been used successfully for a wide variety of flows within the context of astrophysical and geophysical fluid dynamics [3,4]. In these classical model, the stress tensor is assumed to be proportional to the spatial variation of the velocity and then the fluid is referred to as Newtonian fluid. There are, however, many interesting phenomena experimentally observed for various fluid that cannot be described by the Navier–Stokes equations (e.g. see [5] for a brief description of several so-called non-Newtonian features), i.e. the stress tensor depends on the shear rate in a nonlinear manner. One such relation is the stress tensor S depends on the shear rate Du in a polynomial way with p-rate growth, which is firstly proposed by Ladyzhenskaya in [6].

Let $\Omega = (-L, L)^3, L > 0$ is the cube in R^3, $I = (0, T), T > 0$, $Q_T = \Omega \times I$. In this paper, we consider the following nonlinear Boussinesq system

$$\rho\frac{\partial u}{\partial t} + \rho(u\cdot\nabla)u = -\nabla\pi + \operatorname{div}\left((\kappa_1 + \kappa_2|Du|^2)^{\frac{p-2}{2}}Du\right) + \rho e_n\theta, \qquad (x,t)\in Q_T, \tag{1.1}$$

$$\operatorname{div} u = 0, \qquad (x,t)\in Q_T, \tag{1.2}$$

$$\rho\frac{\partial \theta}{\partial t} + \rho(u\cdot\nabla)\theta - \kappa\Delta\theta = g, \qquad (x,t)\in Q_T, \tag{1.3}$$

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where $\rho = const > 0$ is the density; $\kappa = const > 0$ is the thermometric conductivity; κ_1, κ_2 are viscosity coefficients which are positive constants here; e_n is a unit vector in R^n, $1 < p \leq 2$. For the sake of simplicity, we take $\rho = 1, \kappa = 1, \kappa_1 = 1, \kappa_2 = 1$. The unknown functions here are $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)), \theta = \theta(x, t)$, and $\pi = \pi(x, t)$, which stand for the velocity field, the temperature and the pressure of the flow; Du denotes the symmetric part of the velocity gradient ∇u, i.e. $Du = \frac{1}{2}(\nabla u + (\nabla u)^T)$. For a derivation of the nonlinear Boussinesq approximation, we refer the reader to [7,8].

When $p = 2$, problem (1.1)–(1.3) reduces to the well studied classical Boussinesq model(the Newtonian form), which has been analyzed in great extent, see for instance [9–12]. Concerning the more general case, that is both the viscosity and thermal conductivity are temperature dependent, [13] proved existence of reproductivity weak solutions in exterior domains; The existence of regular reproductive solution with Neumann condition on the temperature was studied in [14]; The case of periodic boundary conditions was studied by [15]; The existence of weak and strong solutions of the initial boundary value problem was proved in [16]. Recently, by using the iterative method, [17] obtained a result on existence and uniqueness of strong solutions which is similar to the one proved by [16]. For more related results, we could see also [18–20].

For the non-Newtonian case, the related results are quite few. [21] studied the asymptotic behavior of solution of the Boussinesq approximation where the viscosity depends on polynomial of the shear rate. [22] proved the existence and uniqueness of solution, the existence of attractors for a modified Boussinesq approximations. [23] considered the periodic initial value problem and they proved the existence of the weak solution to (1.1)–(1.3) for $p \geq 2$, its uniqueness and regularity for $p \geq 1 + (2n/(n + 2))$, where n is the space dimension. More recently, for fractional Boussinesq approximation, [24] studied the existence, uniqueness and the long time behavior of the weak solution. Especially, they give the decay estimates of weak solution.

Our goal in this paper is to prove existence, uniqueness and regularity to the solution of system (1.1)–(1.3) with periodic boundary and initial value conditions

$$u(x - e_jL, t) = u(x + e_jL, t), \qquad \theta(x - e_jL, t) = \theta(x + e_jL, t), \tag{1.4}$$

$$u(x, 0) = u_0(x), \qquad \theta(x, 0) = \theta_0(x) \tag{1.5}$$

where $u_0(x), \theta_0(x)$ be given vector valued and scalar valued functions satisfying periodic boundary condition (1.4). We will assume $\int_\Omega g dx = 0$, moreover, in order to ensure that the Poincaré inequality remains valid, we compensate the missing boundary conditions by restricting the solutions to ones with mean value zero.

Before we introduce the notion of a weak solution to (1.1)–(1.5), let us provide some notations and function spaces which will be used in sequence of the paper.

As before let Ω be the cube in R^3. The space $\mathcal{D}(\Omega)$ is defined as the space of functions from $C^\infty(\overline{\Omega})$ with compact support in Ω and $C^\infty_{per}(\Omega)$ denotes the space of functions from $C^\infty(\overline{\Omega})$ which are periodic in Ω. By $(L^q(\Omega), \|.\|_q)$, resp. $(W^{k,p}(\Omega), \|.\|_{k,p})$, we denote the classical Lebesgue and Sobolev spaces. For a Banach space X, we denote by $L^q(I, X)$ the Bochner space with q-integrability and values in X and $C(I, X)$ will stands for the continuous functions with values in X. We use also the following spaces:

$$\mathcal{V} :\equiv \left\{\phi \in C^\infty_{per}(\Omega);\ \mathrm{div}\phi = 0,\ \int_\Omega \phi\, dx = 0\right\},$$

$$\overline{\mathcal{V}} :\equiv \left\{\phi \in C^\infty_{per}(\Omega);\ \int_\Omega \phi\, dx = 0\right\},$$

$$H :\equiv \text{ the closure of } \mathcal{V} \text{ with respect to the } \|\cdot\|_2 - \text{norm},$$

$$V_p :\equiv \text{ the closure of } \mathcal{V} \text{ with respect to the } \|\nabla\cdot\|_p - \text{norm},$$

$$V_\theta :\equiv \text{ the closure of } \mathcal{D} \text{ with respect to the } \|\nabla\cdot\|_2 - \text{norm},$$

$$W^{k,p}_{\mathrm{div}}(\Omega) :\equiv \text{ the closure of } \mathcal{V} \text{ with respect to the } \|\cdot\|_{k,p} - \text{norm},$$

$$W^{k,p}_{per}(\Omega) :\equiv \text{ the closure of } \overline{\mathcal{V}} \text{ with respect to the } \|\cdot\|_{k,p} - \text{norm}.$$

Next, we introduce the notions of weak solutions to (1.1)–(1.5).

Definition 1.1. A couple (u, θ) is called a weak solution of the problems (1.1)–(1.5) if

$$u \in L^p(I, V_p) \cap L^p(I, W^{1,2}_{per}(\Omega)) \cap C(I, H); \qquad \partial_t u \in L^\infty(I, H); \tag{1.6}$$

$$\theta \in L^2(I, V_\theta) \cap C(I, L^2_{per}(\Omega)); \qquad \partial_t \theta \in L^\infty(I, L^2_{per}(\Omega)); \tag{1.7}$$

and for all $\phi \in W^{1,2}(\Omega) \cap V_p,\ \psi \in V_\theta$ the following equality holds for a.e. $t \in I$,

$$\int_\Omega \partial_t u\phi dx + \int_\Omega (u\cdot\nabla)u\phi dx + \int_\Omega (1 + |Du|^2)^{\frac{p-2}{2}} Du : D\phi dx = \int_\Omega e_n\theta\phi dx, \tag{1.8}$$

$$\int_\Omega \partial_t\theta\psi dx + \int_\Omega (u\cdot\nabla)\theta\psi dx + \int_\Omega \nabla\theta\cdot\nabla\psi dx = \int_\Omega g\psi dx. \tag{1.9}$$