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Complete commuting solutions of the Yang–Baxter-like matrix equation for diagonalizable matrices

Qixiang Dong [a], Jiu Ding [b,*]

[a] School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, PR China

[b] Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS 39406, USA

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ABSTRACT

Let A be a square matrix that is diagonalizable. We find all the commuting solutions of the quadratic matrix equation $AXA = XAX$, by taking advantage of the Jordan form structure of A, together with the help of a well-known theorem on the uniqueness of a solution to Sylvester's equation. Two special classes of the given matrix A are further investigated, including circular matrices and those that are equal to some of their powers. Moreover, all the non-commuting solutions are constructed when A is a Householder matrix, based on a spectral perturbation result.

1. Introduction

The classical Yang–Baxter equation was first established by Yang [1] for the delta function Fermi gas in 1967 and then by Baxter [2] for the 8-vertex model in 1972. They showed independently that some rational matrix functions in their studied problems satisfy the nonlinear matrix equation. Since then, more solutions of various forms of the Yang–Baxter equation have been constructed by physicists (see, e.g., [3] and the references therein). In the past three decades this matrix equation has not only been widely studied in statistical physics, but also attracted the attention of mathematicians because of its close relation to such pure mathematics areas as knot theory, braid groups, and quantum groups; see the monographs [4,5].

In its simpler parameter-free format, the original Yang–Baxter equation can be written in the form of the quadratic matrix equation

$$AXA = XAX, \tag{1}$$

where A is a given $n \times n$ matrix. Indeed, by solutions of the Yang–Baxter equation, one means those linear mappings R from the tensor product $V \otimes V$ of a finite dimensional vector space V into itself satisfying

$$R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23},$$

where R_{ij} maps $V \otimes V \otimes V$ into itself such that it acts as R on the two tensor factors i and j and as the identity on the remaining factor. There have been various approaches to solving the Yang–Baxter equation, such as the set-theoretic one [6–8] based on the idea [9] of reducing the problem on the vector space to its basis as a finite set.

* Corresponding author.
E-mail address: jiudin@gmail.com (J. Ding).

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Although some solutions have been found for the Yang–Baxter equation in quantum group theory, no systematical study of (1) has appeared in the literature as a purely linear algebra problem. One possible reason is that solving the equation with a general given matrix A is equivalent to solving a polynomial system of n^2 quadratic equations with n^2 unknowns, which belongs to the central theme of algebraic geometry as a very challenging topic. It seems that the first attempt of solving (1) from the angle of matrix theory is the paper [10] published in 2012, in which Brouwer's fixed point theorem was applied to find a nontrivial solution when A is an invertible row-sum one matrix such that A^{-1} is a stochastic one. Since then, solving the quadratic matrix equation (1), which may be called the *Yang–Baxter-like matrix equation*, has resulted in the publications of [11–15]. But so far, only finitely many solutions have been obtained [12] for a general matrix A, which is based on the spectral theory of matrices. More solutions were obtained in [11,13,15] for various types of matrices A. But all the solutions X of (1) in such papers satisfy the additional property of commutability, in other words, $AX = XA$, with the exception of the paper [11], in which all the solutions of Eq. (1) have been constructed when A is a projection matrix, that is, $A^2 = A$. It is well known that projection matrices are diagonalizable with eigenvalues 0 and 1, except for the zero or identity matrix.

In the present paper we would like to find all the solutions X of the Yang–Baxter-like matrix equation satisfying $AX = XA$. Such solutions are called the *commuting solutions*. This is still a hard problem if A is a general matrix, but with the additional assumption that the matrix A is diagonalizable, we are successful in finding all the commuting solutions of (1), and our main result characterizes the commuting solutions of (1) for diagonalizable matrices. A particular case is that there is a positive integer $k \geq 2$ such that $A^k = A$ and $A^i \neq A$ for $1 \leq i < k$. Of course when $k = 2$, the equation was solved completely in [11]. However, in the more general case of $k > 2$, finding all the solutions is difficult. A consequence of our main result here is that we can find all the commuting solutions in the above special case. A more concrete situation is that A is a Householder matrix so that $A^3 = A$. Not only all the commuting solutions can be easily obtained as a direct application of the main result, but also all the non-commuting solutions are available by means of a spectral perturbation technique.

The approach used here for finding all the commuting solutions of the Yang–Baxter-like matrix equation has first been developed in [14] for finding some or all commuting solutions of Eq. (1) when the matrix A is of several special Jordan forms that had been explored in [13]. The above paper, however, has only constructed the commuting solutions for several individual cases, and consequently no existence conclusions were claimed for general classes of matrices. The achievement of the current paper is that, for any diagonalizable matrix A, all the commuting solutions of the equation have been constructed. This is true if A is, for example, real symmetric, Hermitian, or more generally normal. Many matrices that occur in physical science are diagonalizable, so our complete solution to finding all the commuting solutions of the Yang–Baxter-like matrix equation for the general class of diagonalizable matrices can provide explicit solution expressions for solving such equations in physical applications.

Our main result will be presented in the next section. The applications to two special classes of matrices will be demonstrated in Section 3, in particular we can find all the commuting solutions of (1) when A is a Householder matrix. We further look for all the non-commuting solutions for the case of the Householder matrix in Section 4, An illustrative numerical example will be given in Section 5, and we conclude with Section 6.

2. The commuting solutions

In the following we assume that the matrix A in (1) is *diagonalizable*, that is, A is similar to a diagonal matrix. Let the distinct eigenvalues of A be $\lambda_1, \ldots, \lambda_r$ with each λ_i of multiplicity m_i. Then there is an invertible matrix U such that $U^{-1}AU = D$, where

$$D = \text{diag}(\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r}). \tag{2}$$

Here each I_{m_i} denotes the $m_i \times m_i$ identity matrix.

It is obvious that (see, e.g., [13]) if A is similar to B with similarity matrix V, then any solution of the Yang–Baxter-like matrix equation (1) is similar to a solution of the same equation with A replaced by B, and the similarity matrix is also V. In addition, a solution to the original equation is commuting if and only the corresponding solution to the new equation is commuting. These facts imply that, in order to find all the commuting or non-commuting solutions of (1), it is enough to assume, without loss of generality, that A is already its Jordan canonical form. Thus, in the following we just solve the simplified equation

$$DYD = YDY, \tag{3}$$

where D is given above in (2).

Lemma 2.1. *Let λ be any number and let $D = \lambda I_m$. Then all solutions Y to the Yang–Baxter-like matrix equation (3) are commuting ones and are given by the solutions of the homogeneous equation*

$$\lambda Y(\lambda I - Y) = 0. \tag{4}$$

So, if $\lambda = 0$, then all $m \times m$ matrices Y are solutions of (3), and if $\lambda \neq 0$, then all the solutions of (3) are given by

$$Y = \lambda P,$$

where P are all $m \times m$ projection matrices.