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On a class of nonlinear heat equations with viscoelastic term

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ABSTRACT

The main goal of this paper is to study a model of the strongly nonlinear heat equation with viscoelastic term and nonlinear interior source of the form

$$\begin{cases} \left(1+a|u|^{q-2}\right)u_t - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = f(u), & \text{in } \Omega\times[0,\infty),\\ u=0 & \text{on } \partial\Omega\times[0,\infty),\\ u(x,0)=u_0(x) & \text{in } \Omega. \end{cases}$$

We show the results of local (or global) existence of weak solutions by using the Galerkin approximation method. In addition it has been provided sufficient conditions for the large time decay and the finite time blow-up of the weak solutions.

1. Introduction

Let Ω is an open bounded subset in $\mathbb{R}^n$ with smooth boundary. Consider the following model of the strongly nonlinear heat equation with viscoelastic term and general nonlinear interior source

$$\begin{cases} \left(1+a|u|^{q-2}\right)u_t - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = f(u), \quad \text{in } \Omega\times\mathbb{R}^+,\\ u=0 \quad \text{on } \partial\Omega\times\mathbb{R}^+,\\ u=u_0 \quad \text{on } \Omega, \end{cases} \tag{1.1}$$

where a, q are given positive constants and

$$q\in\left(2,\frac{2n}{n-2}\right) \quad \text{if } n\geq 3 \text{ while } q\in(2,\infty) \text{ if } n=1,2.$$

The initial data is in the energy space, that is $u_0\in H_0^1(\Omega)$. Moreover the relaxation function g and the nonlinear interior source f satisfy some conditions that will be specified later.

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In the physical point of view, this type of problems arise usually in viscoelasticity. In particular, in the absence of nonlinear diffusion term $|u|^{q-2}u_t$, the first equation in (1.1) is reduced to the following equation

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = f(u). \tag{1.2}$$

This equation is the mathematical model of many natural phenomenas in physical science and engineering. For example, in the study of heat conduction in materials with memory, from the heat balance equation the temperature $u(x,t)$ will satisfy Eq. (1.2).

Problems related to Eq. (1.2) have attracted a great deal of attention in last several decades. There have been many results on the existence, blow-up or asymptotic behavior of solutions. For instance in [1], Messaoudi studied Eq. (1.2) in the case $f(u) = b|u|^{p-2}u$, associated with homogeneous Dirichlet boundary condition. If the relaxation function g is assumed to be nonnegative; $g'(t) \le 0$ and

$$\int_0^\infty g(s)ds < \frac{p-2}{p-3/2},$$

he proved the blow-up of weak solution with positive initial energy by the convexity method. We refer to [2–4] for further results on this type of equations.

On the other hand, the first equation in (1.1) without viscoelastic term (that is, the relaxation function g vanishes) can be seen as a special case of doubly nonlinear parabolic-type equations, or the porous medium equations

$$(\varphi(u))_t - \Delta u = f(u), \tag{1.3}$$

if we take $\varphi(u) = u + a|u|^{q-2}u$. We note that in the case of $\varphi(u) = u^m$, by letting $v = \varphi(u) = u^m$, Eq. (1.3) also can be put into the following equation with $\psi(v) = v^{1/m}$

$$v_t - \Delta\psi(v) = f(\psi(v)). \tag{1.4}$$

The blow-up analysis for solutions of quasilinear parabolic systems associated to various boundary conditions has been studied by many authors (see [5–7] and references therein).

From mathematical point of view, problem (1.1) is a generalization of (1.2) and (1.3). The presence of the nonlinear diffusion term $|u|^{q-2}u_t$ caused some difficulties in obtaining the priori estimates. Moreover, we note that the problem (1.1) can be put into the form (1.4) by changing of variable without an explicit representation of ψ. In [8], Polat proved a blow up result for the solution with vanishing initial energy of problem (1.1) when $g = 0$ and $n = 1$. In the case the viscoelastic term is replaced by the term Δu_t, the authors (see [9]) showed the exponential growth of solutions for problem (1.1) with negative or positive initial energy by constructing differential inequalities. This case has been also considered in [10,11].

This paper is the continuation of our previous works [12] in which we showed the sufficient conditions for the exponential growth of weak solution of problem (1.1) when $f(u) = b|u|^{p-2}u$. Our goal is to extend this result and to obtain deeper properties for weak solutions of problem (1.1).

First in Section 3, by using the Galerkin approximation method we are able to prove existence under the natural growth rate of nonlinear source, that is,

$$|f(u)| \le C_1 + C_2|u|^{p-1}, \quad \forall u \in \mathbb{R}.$$

If $q < p < 2+2q/n$ then the solutions only exist locally in time while the global existence will occur in the case of $2 < p \le q$. The main difficulty is overcoming the nonlinear diffusion term. On the other hand, under some restrictions of initial data, we also show that the local in time solutions in the case $q < p < 2 + 2q/n$ can be continued globally in time.

In Section 4, we study the large time decay properties of weak solutions under certain class of initial data and the growth assumptions on the source term. These assumptions are quite general so that they include the function

$$u \mapsto b|u|^{p-2}u + \sum_{i=1}^M \alpha_i|u|^{p_i-2}u, \tag{1.5}$$

with $b > 0$ and all coefficients α_i is nonpositive, as a particular case. Moreover, the decay rate of the solution depends on that rate of the relaxation function g.

Finally, Section 5 deals with the finite time blow-up of weak solutions. The growth assumptions of the nonlinear source are as follows

$$F(u) \le \frac{b}{p}|u|^p, \quad \text{and} \quad uf(u) - (p-\mu)F(u) \ge \widehat{b}|u|^p, \quad \forall u \in \mathbb{R}, \tag{1.6}$$

with $F(u) = \int_0^u f(\xi)d\xi$. Here μ can take any value in the interval $(0, \widehat{p}]$ for some $2 < \widehat{p} < p$; $\widehat{b}$ is a positive constant depending on μ. These type of conditions have been used in several papers for models without viscoelastic term (see [13–15]). In the presence of viscoelastic term the condition, that we imposed on the relaxation function along with (1.6), is new comparing with some other results [16–18]. The blow up results are proved for both cases of initial energy, that is, negative and positive small enough. Here some ideas in [19,3] have been used.