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Robust error estimation in energy and balanced norms for singularly perturbed fourth order problems

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ABSTRACT

We consider singularly perturbed fourth order problems in two dimensions. Under assumptions on the structure of their solutions, we construct layer-adapted meshes and prove for a mixed-method convergence and supercloseness in the associated energy norms as well as convergence in balanced norms.

1. Introduction

We mainly consider the singularly perturbed plate bending problem given by the fourth-order differential equation

$$\varepsilon^2\Delta^2 u - b\Delta u + (c\cdot\nabla)u + du = f \quad \text{in } \Omega := (0,1)^2, \tag{1a}$$

where $b \ge b_0 > 1$, $d - \frac{1}{2}(\operatorname{div} c + \Delta b) \ge \delta > 0$ and $f \in L^2(\Omega)$ are smooth functions, with the boundary conditions

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma := \partial\Omega. \tag{1b}$$

This is a model for the so-called clamped plate. With the usual weak formulation we immediately have a unique solution $u \in H_0^2(\Omega)$. But a conforming finite element discretisation requires C^1-elements, see i.e. [1], which are less popular in 2d. Therefore, mixed or non-conforming methods are widely used. In this paper we want to study mixed methods.

Using in the non-singularly perturbed case ($\varepsilon = 1$) a p-th order finite-element approximation for u and $w = \Delta u$ results in an error estimate

$$\|u - u_h\|_1 + h\|w - w_h\|_0 \le Ch^p\|u\|_{p+1} \tag{2}$$

for the discrete solutions u_h and w_h on a standard shape-regular mesh with $p \ge 2$, see, for instance, [2,3]. We use the standard notation of Sobolev spaces, where $\|\cdot\|_0$ is the L^2-norm, $|\cdot|_k$ the seminorm in H^k and $\|\cdot\|_k$ the full H^k-norm. Furthermore, we denote by $\langle u, v\rangle_D$ the L^2-scalar product over a domain $D \subset \Omega$. If $D = \Omega$ we drop the subscript. In the singularly perturbed

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case the boundary layers of problem (1), even if the layers are weak, lead to $\|u\|_{p+1} \to \infty$ for $\varepsilon \to 0$ and $p \geq 2$. Thus, the estimate (2) becomes worthless in this scenario.

In this paper we use layer-adapted meshes with a number of cells of order N^2. Then we are able to prove for problem (1) with $b > 0$ the robust error estimate (see Section 3)

$$\|u - u_h\|_1 + \varepsilon\|w - w_h\|_0 \leq C(\varepsilon^{1/2} + N^{-1}\ln N)(N^{-1}\ln N)^{p-1}, \tag{3}$$

where here and further on C denotes a generic constant independent of ε and N. This result can be improved using supercloseness properties for $\mathcal{Q}_p$-elements to robust p-th order convergence. Finally, we also estimate the error in a so-called balanced norm.

For non-conforming methods, some robust error estimates on standard meshes are known [4–7] but characterised by a low order (1/2). Layer-adapted meshes are to our knowledge only investigated in combination with the C^0-interior penalty method [8], resulting in an error estimate similar to (3). For our mixed method however, we can additionally prove supercloseness for $\mathcal{Q}_p$-elements and estimate in a balanced norm. Both of these results are open questions for the C^0-interior penalty method.

In Section 4 we apply the techniques of Section 3 to two other problems. First we investigate a problem modelling the simply supported plate: Eq. (1a) together with the boundary conditions

$$\tilde{u} = \Delta\tilde{u} = 0 \quad \text{on } \Gamma. \tag{1c}$$

In Section 4.2 we consider the "reaction–diffusion-type" problem

$$\varepsilon^2\Delta\check{u} + d\check{u} = f \quad \text{in } \Omega$$

with $d \geq \delta > 1$ and the boundary conditions (1b) or (1c). This problem is characterised by strong boundary layers and we have to modify the techniques from Section 3.

If $b \geq b_0 > 0$, in a rectangular domain we have $u \in H_0^2 \cap H^4$, see [9] and $\tilde{u} \in H_0^1 \cap H^4$, see [10]. The same holds for $\check{u}$ if $d \geq \delta > 0$.

2. Solution decomposition and meshes

Asymptotic expansions for higher-order elliptic singularly perturbed problems are known in smooth domains [11]. In this case only boundary layers exist. From second-order problems it is known, that in domains with corners additional corner layers arise. The proof of the existence of solutions of the corner-layer equations for reaction–diffusion problems is extremely complicated, see [12, Section 3.2]. For higher-order problems this is an open problem.

We simply prescribe the layer structure analogously to the results for second-order equations and assume (requiring additional smoothness of the solution and the data) that the solution u of problem (1a) + (1b) can be decomposed into a smooth part, boundary layers and corner layers:

$$u = S + E_1 + E_2 + E_3 + E_4 + E_{12} + E_{23} + E_{34} + E_{41} = S + \sum_{k\in\mathcal{I}} E_k, \tag{4}$$

where $\mathcal{I} = \{1, 2, 3, 4, 12, 23, 34, 41\}$. Here S stands for the smooth part, E_k with $k = 1, 2, 3, 4$ for a boundary layer and E_k with $k = 12, 23, 34, 41$ for a corner layer. More precisely, we assume

$$\|\partial_x^i\partial_y^j S\|_0 \leq C, \qquad |\partial_x^i\partial_y^j E_1(x, y)| \leq C\varepsilon^{1-i}\mathrm{e}^{-x/\varepsilon},$$
$$|\partial_x^i\partial_y^j E_2(x, y)| \leq C\varepsilon^{1-j}\mathrm{e}^{-y/\varepsilon}, \qquad |\partial_x^i\partial_y^j E_{12}(x, y)| \leq C\varepsilon^{1-i-j}\mathrm{e}^{-x/\varepsilon}\mathrm{e}^{-y/\varepsilon},$$

and similarly for the other components of the decomposition. Similarly, we assume a decomposition for $\tilde{u}$ with

$$\tilde{u} = \tilde{S} + \sum_{i\in\mathcal{I}} \tilde{E}_i \tag{5}$$

where

$$\|\partial_x^i\partial_y^j \tilde{S}\|_0 \leq C, \qquad |\partial_x^i\partial_y^j \tilde{E}_1(x, y)| \leq C\varepsilon^{2-i}\mathrm{e}^{-x/\varepsilon},$$
$$|\partial_x^i\partial_y^j \tilde{E}_2(x, y)| \leq C\varepsilon^{2-j}\mathrm{e}^{-y/\varepsilon}, \qquad |\partial_x^i\partial_y^j \tilde{E}_{12}(x, y)| \leq C\varepsilon^{2-i-j}\mathrm{e}^{-x/\varepsilon}\mathrm{e}^{-y/\varepsilon},$$

and similarly for the other components of the decomposition.

Remark 2.1. In the following numerical analysis we need the estimates from above assumptions in a $L^2(D)$-sense for non-empty subdomains $D \subset \Omega$ for $i + j = p + 1$ and in the supercloseness analysis additionally for $i = 1, 2$ and $j = p + 1$ as well as $i = 1$ and $j = p + 2$ and the pairs with interchanged i and j.

The pointwise estimates for the layer components are essentially only needed for $i + j = 0, 1$ and in the supercloseness analysis additionally for $i = 1$ and $j = p + 1$, and $i = p + 1$ and $j = 1$.