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Solutions of some quadratic Diophantine equations

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ABSTRACT

In this study, we investigate positive integer solutions of the Diophantine equations $x^2 - kxy \mp y^2 \mp x = 0$ and $x^2 - kxy - y^2 \mp y = 0$. It is shown that when $k > 3$, $x^2 - kxy + y^2 + x = 0$ has no positive integer solutions but the equation $x^2 - kxy + y^2 - x = 0$ has positive integer solutions. Moreover, it is shown that the equations $x^2 - kxy - y^2 \mp x = 0$ and $x^2 - kxy - y^2 \mp y = 0$ have positive solutions when $k \geq 1$.

1. Introduction

In [1], the authors dealt with the equation

$$x^2 - kxy + y^2 + x = 0 \tag{1.1}$$

and they showed that Eq. (1.1) has no positive integer solutions x and y when $k > 3$ but Eq. (1.1) has an infinite number of positive integer solutions x and y when $k = 3$. When $k = 3$, they gave a process giving an infinite set of positive solutions of Eq. (1.1). Moreover, they guess that when $k = 3$, all the positive integer solutions of Eq. (1.1) are of the form $(x, y) = (F_{2n-1}^2, F_{2n-1}F_{2n-3})$ or $(x, y) = (F_{2n-1}^2, F_{2n-1}F_{2n+1})$ where F_n is the nth Fibonacci number.

In this study, we consider the equations

$$x^2 - kxy + y^2 \mp x = 0, \tag{1.2}$$

$$x^2 - kxy - y^2 \mp x = 0, \tag{1.3}$$

and other similar equations. Firstly, we will find all positive integer solutions of the equation

$$x^2 - 3xy + y^2 \mp x = 0.$$

Then we will find all positive integer solutions of the equations

$$x^2 - xy - y^2 \mp 5x = 0 \tag{1.4}$$

and

$$x^2 - 3xy + y^2 \mp 5x = 0. \tag{1.5}$$

Moreover, we will find all positive integer solutions of the equations

$$x^2 - xy - y^2 \mp x = 0$$

and

$$x^2 - xy - y^2 \mp y = 0.$$

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Lastly, when $k > 3$, we will show that Eq. (1.1) has no positive integer solutions but the equation $x^2 - kxy + y^2 - x = 0$ has positive integer solutions. Moreover, we will show that the equations $x^2 - kxy - y^2 \mp x = 0$ and $x^2 - kxy - y^2 \mp y = 0$ have positive solutions when $k \geq 1$.

Solutions of some of the above equations are related to the Fibonacci numbers. Now we briefly mention the Fibonacci sequence $\{F_n\}$. The Fibonacci sequence $\{F_n\}$ is defined by $F_0 = 0$, $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. F_n is called the nth Fibonacci number. Fibonacci numbers for negative subscripts are defined as $F_{-n} = (-1)^n F_n$ for $n \geq 1$. It is well known that $F_{n+1} = F_n + F_{n-1}$ for every $n \in \mathbb{Z}$. For more information about Fibonacci sequence one can consult [2,3]. Let α and β denote the roots of the equation $x^2 - x - 1 = 0$. Then $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = (1 - \sqrt{5})/2$. It can be seen that $\alpha\beta = -1$ and $\alpha + \beta = 1$. Moreover it is well known and easy to show that

$$\alpha^n = \alpha F_n + F_{n-1} \tag{1.6}$$

and

$$\beta^n = \beta F_n + F_{n-1}$$

for every $n \in \mathbb{Z}$. On the other hand, it can be shown by induction that

$$F_n^2 - F_n F_{n-1} - F_{n-1}^2 = (-1)^{n+1} \tag{1.7}$$

for every $n \in \mathbb{Z}$. Now we give a theorem from [4], which is related to the set of the units of the ring $\mathbb{Z}[\alpha]$ where $\mathbb{Z}[\alpha] = \{a\alpha + b : a, b \in \mathbb{Z}\}$.

Theorem 1.1. *The set of the units of* $\mathbb{Z}[\alpha]$ *is* $\{\mp\alpha^n : n \in \mathbb{Z}\}$.

Using the above theorem and identity (1.7), we can give the following theorem. The proof of the theorem can be found in [3] but for the sake of completeness, we will give its proof in a different way.

Theorem 1.2. *All positive integer solutions of the equation* $x^2 - xy - y^2 = \mp 1$ *are given by* $(x, y) = (F_n, F_{n-1})$ *with* $n \geq 2$.

Proof. If $(x, y) = (F_n, F_{n-1})$, then from identity (1.7), it follows that $x^2 - xy - y^2 = \mp 1$. Now assume that $x^2 - xy - y^2 = \mp 1$ for some positive integers x and y. Then $\alpha x + y > 1$ and $(\alpha x + y)(\beta x + y) = \mp 1$. This shows that $\alpha x + y$ is a unit in $\mathbb{Z}[\alpha]$. Then, by Theorem 1.1, $\alpha x + y = \alpha^n$ for some $n \geq 2$. Thus $\alpha x + y = \alpha F_n + F_{n-1}$, which implies that $(x, y) = (F_n, F_{n-1})$. $\square$

Corollary 1.3. *All positive integer solutions of the equation* $x^2 - xy - y^2 = 1$ *are given by* $(x, y) = (F_{2n+1}, F_{2n})$ *with* $n \geq 1$.

Corollary 1.4. *All positive integer solutions of the equation* $x^2 - xy - y^2 = -1$ *are given by* $(x, y) = (F_{2n}, F_{2n-1})$ *with* $n \geq 1$.

Theorem 1.5. *All positive integer solutions of the equation* $x^2 - 3xy + y^2 = 1$ *are given by* $(x, y) = (F_{2n+2}, F_{2n})$ *with* $n \geq 1$.

Proof. Assume that $x^2 - 3xy + y^2 = 1$ for some positive integers x and y. Without loss of generality we may suppose that $x > y$. Then $(x-y)^2 - y(x-y) - y^2 = x^2 - 3xy + y^2 = 1$ and therefore $(x - y, y) = (F_{2n+1}, F_{2n})$ with $n \geq 1$, by Corollary 1.3. That is, $x - y = F_{2n+1}$ and $y = F_{2n}$. This implies that $x = F_{2n+1} + y = F_{2n+1} + F_{2n} = F_{2n+2}$. Therefore $(x, y) = (F_{2n+2}, F_{2n})$ with $n \geq 1$. Conversely, if $(x, y) = (F_{2n+2}, F_{2n})$, then using the identity $F_{2n+2} = F_{2n} + F_{2n-1}$ and identity (1.7) one can show that $x^2 - 3xy + y^2 = 1$. $\square$

The proof of the following theorem is similar and therefore we omit it.

Theorem 1.6. *All positive integer solutions of the equation* $x^2 - 3xy + y^2 = -1$ *are given by* $(x, y) = (F_{2n+1}, F_{2n-1})$ *with* $n \geq 0$.

2. Main theorems

In this section we consider Eqs. (1.2)–(1.4), and other similar equations. Now we give the following theorem (see [1] for another proof).

Theorem 2.1. *If positive integers* x, y, k *satisfy Eq.* (1.2) *or* (1.3), *then* $x = u^2$ *and* $y = uv$ *for some positive integers* u *and* v.

Proof. Assume that positive integers x, y, k satisfy Eq. (1.2). Then it follows that $x|y^2$ and therefore $y^2 = xz$ for some positive integer z. Assume that $p|x$ and $p|z$ for some prime number p. Then $p|y$ and from (1.2), it is seen that $x - ky + z \mp 1 = 0$. This implies that $p|1$, which is a contradiction. Therefore $\gcd(x, z) = 1$. This shows that $x = u^2$ and $z = v^2$ for some positive integers u and v. Then $y^2 = xz = u^2v^2 = (uv)^2$ and therefore $y = uv$. $\square$

Theorem 2.2. *All positive integer solutions of the equation* $x^2 - 3xy + y^2 + x = 0$ *are given by* $(x, y) = (F_{2n+1}^2, F_{2n+1}F_{2n-1})$ *or* $(x, y) = (F_{2n-1}^2, F_{2n-1}F_{2n+1})$ *with* $n \geq 0$.

Proof. Assume that $x^2 - 3xy + y^2 + x = 0$ for some positive integers x and y. Then by Theorem 2.1, $x = u^2$ and $y = uv$ for some positive integers u and v. Then it follows that $u^4 - 3u^3v + u^2v^2 + u^2 = 0$, which implies that $u^2 - 3uv + v^2 + 1 = 0$. That is, $u^2 - 3uv + v^2 = -1$. By Theorem 1.6, it follows that $(u, v) = (F_{2n+1}, F_{2n-1})$ or $(u, v) = (F_{2n-1}, F_{2n+1})$ with $n \geq 0$. This shows that $(x, y) = (F_{2n+1}^2, F_{2n+1}F_{2n-1})$ or $(x, y) = (F_{2n-1}^2, F_{2n-1}F_{2n+1})$ with $n \geq 0$. Conversely, if $(x, y) = (F_{2n+1}^2, F_{2n+1}F_{2n-1})$ or $(x, y) = (F_{2n-1}^2, F_{2n-1}F_{2n+1})$ with $n \geq 0$, then from Theorem 1.6, it follows that $x^2 - 3xy + y^2 + x = 0$. $\square$