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The existence of solutions for a fractional multi-point boundary value problem☆

Zhanbing Bai *, Yinghan Zhang

College of Information Science and Engineering, Shandong University of Science and Technology, Qingdao 266510, PR China

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A multi-point boundary value problem for a fractional ordinary differential equation is considered in this paper. An existence result is obtained with the use of the coincidence degree theory.

1. Introduction

In this paper, we study the multi-point boundary value problem

$$D_{0+}^{\alpha}u(t)=f(t,u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha-2}u(t)),\quad t\in(0,1), \tag{1.1}$$

$$I_{0+}^{3-\alpha}u(0)=0,\ D_{0+}^{\alpha-1}u(0)=D_{0+}^{\alpha-1}u(\eta),\ u(1)=\sum_{i=1}^{m}\alpha_i u(\eta_i), \tag{1.2}$$

where $2<\alpha\le 3$, $0<\eta\le 1$, $0<\eta_1<\cdots<\eta_m<1$, $m\ge 2$, and $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ satisfying the Carathéodory conditions. D_{0+}^{α} and I_{0+}^{α} are the standard Riemann–Liouville fractional derivative and fractional integral respectively, and

$$\sum_{i=1}^{m}\alpha_i\eta_i^{\alpha-1}=\sum_{i=1}^{m}\alpha_i\eta_i^{\alpha-2}=1. \tag{1.3}$$

We assume, in addition, that

$$R=\frac{1}{\alpha}\eta^{\alpha}\frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left[1-\sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-2}\right]-\frac{1}{\alpha-1}\eta^{\alpha-1}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\left[1-\sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-1}\right]\neq 0, \tag{1.4}$$

where Γ is the Gamma function. Due to the condition (1.3) the fractional differential operator in (1.1) is not invertible.

The subject of fractional calculus has gained considerable popularity and importance due to its frequent appearance in various fields such as physics, chemistry, and engineering. Many methods have been introduced for solving fractional

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* Corresponding author.
E-mail addresses: zhanbingbai@163.com (Z. Bai), zhangyinghan007@126.com (Y. Zhang).

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differential equations, such as the popular Laplace transform method, the iteration method, the Fourier transform method and the operational method. For details, see [1,2] and the references therein.

It should be noted that most of the papers and books on fractional calculus are devoted to the solvability of linear initial value fractional differential equations in terms of special functions [1–3]. Recently, there have been some papers dealing with the existence and multiplicity of solutions (or positive solutions) of nonlinear initial value fractional differential equations by the use of techniques of nonlinear analysis (fixed point theorems, Leray–Schauder theory, etc.); see e.g. [4–6].

However, there are few papers that consider the boundary value problem at resonance for nonlinear ordinary differential equations of fractional order. In [5], Bai investigated the nonlinear nonlocal problem

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad 0 < t < 1,$$
$$u(0) = 0, \qquad \beta u(\eta) = u(1),$$

where $1 < \alpha \le 2$, $0 < \beta\eta^{\alpha-1} < 1$. But there are few papers that consider the case $\beta\eta^{\alpha-1} = 1$. In the recent paper [7], we considered the existence of solutions of the fractional ordinary differential equation

$$D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)) + e(t), \quad 0 < t < 1,$$

subject to the following boundary value conditions:

$$I_{0+}^{2-\alpha}u(0) = 0, \qquad u(1) = \sigma u(\eta),$$

or

$$I_{0+}^{2-\alpha}u(0) = 0, \qquad D_{0+}^{\alpha-1}u(1) = D_{0+}^{\alpha-1}u(\eta),$$

where $1 < \alpha \le 2$ is a real number, D_{0+}^{α} and I_{0+}^{α} are the standard Riemann–Liouville derivative and integral respectively, and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $e(t) \in L^1[0, 1]$, $\sigma \in (0, \infty)$ and $\eta \in (0, 1)$ are given constants such that $\sigma\eta^{\alpha-1} = 1$. By these conditions, the kernel of the linear operator $L = D_{0+}^{\alpha}$ is one dimensional. It is therefore natural to consider the case of two dimensions. Integer-order boundary value problems at resonance have been studied by many authors [8–17], but there are few papers that consider the fractional-order boundary problems at resonance. So, motivated by the above work and recent studies on fractional differential equations [18–27], in this paper we consider the existence of solutions for a nonlinear fractional multi-point boundary value problem at resonance.

The purpose of this paper is to study the existence of a solution for boundary value problem (1.1), (1.2) at resonance, and establish an existence theorem under a nonlinear growth restriction on f. Our method is based upon the coincidence degree theory of Mawhin [28].

Now, we will briefly recall some notation and an abstract existence result.

Let Y, Z be real Banach spaces, $L : \mathrm{dom}(L) \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y, Q : Z \to Z$ be continuous projectors such that $\mathrm{Im}(P) = \mathrm{Ker}(L)$, $\mathrm{Ker}(Q) = \mathrm{Im}(L)$ and $Y = \mathrm{Ker}(L) \oplus \mathrm{Ker}(P)$, $Z = \mathrm{Im}(L) \oplus \mathrm{Im}(Q)$. It follows that $L|_{\mathrm{dom}(L)\cap\mathrm{Ker}(P)} : \mathrm{dom}(L) \cap \mathrm{Ker}(P) \to \mathrm{Im}(L)$ is invertible. We denote the inverse of the map by K_P. If Ω is an open bounded subset of Y such that $\mathrm{dom}(L) \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to Y$ is compact.

The theorem that we used is Theorem 2.4 of [28].

Theorem 1.1. *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\mathrm{dom}(L) \setminus \mathrm{Ker}(L)) \cap \partial\Omega] \times (0, 1)$;*
(ii) *$Nx \notin \mathrm{Im}(L)$ for every $x \in \mathrm{Ker}(L) \cap \partial\Omega$;*
(iii) *$\deg\left(JQN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), 0\right) \neq 0$, where $Q : Z \to Z$ is a projection as above with $\mathrm{Im}(L) = \mathrm{Ker}(Q)$ and $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$ is any isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$.

The rest of this paper is organized as follows. In Section 2, we give some notation and lemmas. In Section 3, we establish a theorem of existence of a solution for the problem (1.1), (1.2).

2. Background materials and preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions for fractional calculus theory. These definitions can be found in the recent literature [2,5,6].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s$$

provided the right side is pointwise defined on $(0, \infty)$. And we let $I_{0+}^{0}y(t) = y(t)$ for every continuous $y : (0, \infty) \to \mathbb{R}$.