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A complete characterization of nonlinear absorption for the evolution p-Laplacian equations to have positive or extinctive solutions

Soon-Yeong Chung [a], Jea-Hyun Park [b,*]

[a] Department of Mathematics and Program of Integrated Biotechnology, Sogang University, Seoul 121-742, Republic of Korea

[b] Department of Mathematics, Kunsan National University, Kunsan 573-701, Republic of Korea

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ABSTRACT

This paper is concerned with long time behaviors of solutions to the initial–boundary value problem of the evolution p-Laplacian equations with nonlinear absorption. A long time behavior of solutions to these equations has been studied so far for particular types of nonlinear absorption term. The purpose of this paper is to give a complete characterization of the nonlinear absorption term, according to the parameter p in p-Laplacian operator and the growth of the nonlinear absorption term near the origin, in order to determine whether the solution to the equations is extinctive or positive. In addition, we also give upper bounds for extinction times of solutions.

1. Introduction

In this paper, we investigate extinction and positivity properties for weak solutions to the evolution p-Laplacian equations with a non-homogeneous term

$$\frac{\partial u}{\partial t}(x,t) = \Delta_p u(x,t) - f(u(x,t)), \quad (x,t) \in \Omega \times (0,\infty), \tag{1}$$

subject to the initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{3}$$

where $p > 1$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, moreover, we assume that the initial data u_0 is a non-negative function in $L^\infty(\Omega)$ satisfying that $\inf_{x\in\Omega_0} u_0(x) > 0$ for some subset $\Omega_0 \subset \Omega$ of a nonzero measure, (we simply say u_0 is non-trivial), and the non-homogeneous term $f : [0,\infty) \to \mathbb{R}$ is continuous and satisfies $f(0) = 0$, $f(u) > 0$ for all $u > 0$.

Here, the definition of weak solution to Eqs. (1)–(3) is as follows:

Definition 1. A function u is said to be a weak solution of Eqs. (1)–(3), if u satisfies the following conditions:

$$u \in L^\infty(\Omega_\infty) \cap L^p\big(0,\infty; W^{1,p}(\Omega)\big), \quad \frac{\partial u}{\partial t} \in L^2(\Omega_\infty),$$

* Corresponding author.
E-mail addresses: sychung@sogang.ac.kr (S.-Y. Chung), parkjhm@kunsan.ac.kr (J.-H. Park).

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and for each $T > 0$,

$$\int_{\Omega} u(x,T)\varphi(x,T)dx - \int_{\Omega} u_0(x)\varphi(x,0)dx = \int_0^T \int_{\Omega} \left(u\frac{\partial \varphi}{\partial t} - |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi - f(u)\varphi \right) dxdt \tag{4}$$

for all $\varphi \in L^p\big(0,T; W_0^{1,p}(\Omega)\big)$ with $\frac{\partial \varphi}{\partial t} \in L^2(\Omega_T)$ where $\Omega_T := \Omega \times (0,T)$ and $\Omega_\infty := \Omega \times (0,\infty)$.

What we are interested in is a complete characterization of the nonhomogeneous term f to distinguish two cases, (extinction and positivity) for the weak solution to the evolution p-Laplacian equations (1)–(3) with, in addition, an assumption for the non-homogeneous term f that

(H) for each compact subset $K \subset [0,\infty)$, there exists $\alpha_K \in \mathbb{R}$ such that

$$f(u) - f(v) \geq \alpha_K (u - v), \quad \text{for all } u > v \in K.$$

The extinction and positivity(or non-extinction) properties of solutions are one of the most important features of evolution equations. It has been intensively researched during the past few decades. But in many cases, researchers have considered special cases about the non-homogeneous term f. For examples, the case that $f(u) = u^q, q > 0$ is called ‘cool source’, the case that $f(u) = -u^q, q > 0$ is called ‘hot source’, and the case that $f(u) = \int_\Omega u^q(y,t)dy$ is called ‘nonlocal source’.

In [1], Kalashnikov considered the Laplacian case (i.e. $p = 2$) and $\Omega = \mathbb{R}^1$, i.e. the semilinear Cauchy problem:

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) - f(u(x,t)), \quad (x,t) \in \mathbb{R}^1 \times (0,\infty),$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^1.$$

He proved that if the non-homogeneous term f satisfies

$$\int_0^1 \frac{ds}{f(s)} < \infty,$$

then the bounded, nonnegative solution to the semilinear Cauchy problem must vanish after some finite time $T > 0$ and if the non-homogeneous term f satisfies

$$\int_0^1 \frac{ds}{(sf(s))^{1/2}} < \infty, \tag{5}$$

then the solution with compact support initially has compact support at all time $t > 0$.

In 1979, Evans and Knerr [2] showed that in the N dimensional case, if u_0 merely goes to zero uniformly as $|x| \to \infty$ and f satisfies (5), then for each $t > 0$, the support of solution to the semilinear Cauchy problem is bounded and the solution is extinctive after finite time $T > 0$. For other results in the case where $p = 2$, see [3,4] and references therein.

In [5], Lair and Oxley discussed the following equations

$$\frac{\partial u}{\partial t}(x,t) = \Delta \varphi(u(x,t)) - f(u(x,t)), \quad (x,t) \in \Omega \times (0,\infty),$$
$$u(x,0) = u_0(x) \in C(\overline{\Omega}) \cap L^1(\Omega), \quad x \in \Omega,$$
$$\varphi(u(x,t)) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty),$$

where φ is nondecreasing, nonnegative $C^1((0,\infty)) \cap C([0,\infty))$ function satisfying $\varphi(0) = 0$. They proved that if the function φ' is defined and bounded on a finite interval, then the generalized solution to the above equation has finite extinction time if and only if

$$\int_0^\epsilon \frac{ds}{f(s)} < \infty.$$

(For the definition of the generalized solution, please see the paper). In [6], Ning gave some characterization of non-homogeneous term f to have extinctive or non-extinctive weak solutions to degenerate parabolic equations with Neumann boundary data combined with Dirichlet data

Note that as far as the authors know, there are almost no results for the extinctive or positive properties of weak solution to the evolution p-Laplacian equations (1)–(3). On the other hand, the special cases of the non-homogeneous term f, (for examples, cool source, hot source, nonlocal source and so on) have been actively discussed and studied.

In [7], Gu proved that if $p \in (1,2)$ or $q \in (0,1)$, then the solution to the following equation:

$$\frac{\partial u}{\partial t}(x,t) = \Delta_p u(x,t) - au^q(x,t), \quad (x,t) \in \Omega \times (0,\infty), \tag{6}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \quad \text{and} \quad u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{7}$$