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# Analysis of velocity-components decoupled projection method for the incompressible Navier–Stokes equations

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ABSTRACT

We study the temporal accuracy and stability of the velocity-components decoupled projection method (VDPM) for fully discrete incompressible Navier–Stokes equations. In particular, we investigate the effect of three formulations of the nonlinear convection term, which include the advective, skew-symmetric, and divergence forms, on the temporal accuracy and stability. Second-order temporal accuracy of the VDPM for both velocity and pressure is verified by establishing global error estimates in terms of a discrete $l^2$-norm. Considering the energy evolution, we demonstrate that the VDPM is stable when the time step is less than or equal to a constant. Stability diagrams, which display the distributions of the maximum magnitude of the eigenvalues of the corresponding amplification matrices, are obtained using von Neumann analysis. These diagrams indicate that the advective form is more stable than the other formulations of the nonlinear convection term. Numerical tests are performed in order to support the mathematical findings involving temporal accuracy and stability, and the effects of the formulations of the nonlinear convection term are analyzed. Overall, our results indicate that the VDPM along with an advective discrete convection operator is almost unconditionally stable, second-order accurate in time, and computationally efficient because of the non-iterative solution procedure in solving the decoupled momentum equations.



## 1. Introduction

Various projection methods have been widely studied and used for time-dependent incompressible Navier–Stokes equations [1–11]. The original projection method was first proposed by Chorin [3]. It uses Hodge decomposition for pressure-velocity decoupling and an implicit treatment of the nonlinear convection and linear diffusion terms. The method is simple but first-order accurate in time and second-order accurate in space for a periodic box in two-dimensional (2D) and three-dimensional (3D) domains. Subsequently, based on Chorin's method [3], Kim and Moin [7] proposed a semi-implicit projection method (SIPM), which replaces the treatment of the nonlinear convection term with a second-order explicit Adams–Bashforth scheme and provides improved boundary conditions for the intermediate velocity. This, in turn, respectively produces second-order temporal accuracy for velocity and first-order accuracy for pressure in the periodic domains, and first-order temporal accuracy for velocity and pressure in the general domains [9]. In order to resolve the

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issues related to the intermediate velocity boundary conditions in [7], Perot [9] modified Kim and Moin's method using the generalized block lower–upper (LU) decomposition of fully discrete Navier–Stokes equations. Here, the solving process does not require boundary conditions for intermediate velocity and pressure. The method still provides second-order temporal accuracy for velocity but first-order temporal accuracy for pressure. When pressure is evaluated at half-integer time levels, second-order temporal accuracy is achieved [12].

Furthermore, in order to obtain second-order temporal accuracy for both velocity and pressure, van Kan [11] proposed a pressure-correction projection method that employs an alternating direction implicit (ADI) formulation of the nonlinear convection and linear diffusion terms, which are discretized using the Crank–Nicolson scheme. In this method, pressure is explicitly treated in the first step, while velocity is corrected using a pressure increment in the second step after a new pressure has been obtained by solving a discrete Poisson equation. As the dual of the pressure-correction projection method, Guermond and Shen [6] introduced a velocity-correction projection method, which switches the roles of velocity and pressure in the pressure-correction projection method. Choi and Moin [2] developed a fully implicit projection method, in which both nonlinear convection and linear diffusion terms are advanced using the Crank–Nicolson scheme in time. Even though this method admits larger computational time steps for stable numerical solutions, it requires an inevitable iterative procedure in order to determine the intermediate velocity.

In order to address fully discrete Navier–Stokes equations with the Crank–Nicolson scheme, Kim et al. [8] linearized the nonlinear convection term and used a block LU decomposition along with approximate factorization, similar to the approach in [9]. This enabled all primitive variables, such as velocity components and pressure, to be decoupled. This method produced the decoupled linear system for intermediate velocity, without requiring an iterative procedure. Furthermore, the authors numerically demonstrated that second-order temporal accuracy is preserved using the approximate factorization and without modifying the boundary conditions. Their results also suggested that this method overcame the Courant–Friedrichs–Lewy (CFL) number restriction. Moreover, this method's computation time per time step is comparable to that of a semi-implicit projection method. Henceforth, the method is called the velocity-components decoupled projection method (VDPM).

Over the last three decades, projection methods have been mathematically analyzed in terms of temporal accuracy and stability based on linear Stokes equations [5,12–17] or Navier–Stokes equations [11,18–20]. For example, van Kan [11] demonstrated the second-order temporal accuracy of the pressure-correction projection method with the Crank–Nicolson scheme using a system of constrained ordinary differential equations, which are similar to the fully discrete Navier–Stokes equations in discrete time and space, under reasonably weak assumptions. In addition, the method for a linearized problem was shown to be unconditionally stable in the sense of Lyapunov functional [11]. Moreover, Shen [20] provided an error analysis of the pseudo-compressibility projection method, which introduces a pressure stabilizing or regularizing term in the equation of mass conservation. Applying the Crank–Nicolson scheme to a semi-discrete perturbed system, which can be viewed as an approximation of the incompressible Navier–Stokes equations with a skew-symmetric discrete convection operator, he confirmed that the projection method is second-order accurate for velocity but only first-order accurate for pressure in time.

Subsequently, Brown et al. [12] performed a normal mode analysis of semi-discrete Stokes equations, which is a system with discrete time and continuous space, in periodic domains. They demonstrated that the projection methods [1,7,9] are second-order accurate for velocity but typically just first-order accurate for pressure in time. In addition, they proposed a fully second-order accurate projection algorithm based on the global pressure-update formula with specified numerical boundary conditions. Guy and Fogelson [16] applied the normal mode analysis [12] to fully discrete Stokes equations in order to investigate the theoretical stability of the projection methods [1,7,9]. Based on the analysis of a one-dimensional model problem that represents time-dependent Stokes equations with homogeneous boundary conditions, they determined that all the projection methods [1,7,9] are unconditionally stable on the marker-and-cell (MAC) mesh [21]. However, one of these projection methods (PmII in [12]) on the cell-centered mesh is susceptible to numerical instabilities resulting from the pressure gradient near the boundary.

The VDPM proposed by Kim et al. [8], in which all primitive variables are decoupled as in the SIPM, has been widely used both efficiently and feasibly for various fluid mechanics problems [22–26]. They demonstrated that the method is temporally second-order accurate for both velocity and pressure, and almost unconditionally stable based on the numerical simulations of some benchmark problems. To the best of our knowledge, no previous semi-implicit projection method is more stable than the VDPM. However, a rigorous mathematical analysis does not exist within the literature for the temporal accuracy and stability of the VDPM. Moreover, the effect of the formulation or treatment of the nonlinear convection term on temporal accuracy, as well as stability, remains unclear.

Our objective is to provide a mathematical justification for the temporal accuracy and stability of the velocity-components decoupled projection method (VDPM) proposed in [8] by considering the time-dependent incompressible Navier–Stokes equations with the Crank–Nicolson scheme. Using the linearization of the nonlinear convection term and a block LU decomposition along with approximate factorization, the VDPM enables to decouple the velocity components and pressure. In order to prove a second-order accuracy for velocity and pressure with respect to time, we consider the temporal changes of the global errors between the VDPM and the Crank–Nicolson solutions in a discrete $l^2$-norm under reasonably weak assumptions. We use the energy estimation based on the fully discrete Navier–Stokes equations for the stability analysis. Moreover, we apply the von Neumann analysis to the linearized Navier–Stokes equations in order to obtain stability diagrams that represent the distributions of the maximum magnitude of the eigenvalues of the corresponding amplification