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The Laplace transform on isolated time scales

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ABSTRACT

Starting with a general definition of the Laplace transform on arbitrary time scales, we specify the Laplace transform on isolated time scales, prove several properties of the Laplace transform in this case, and establish a formula for the inverse Laplace transform. The concept of convolution is considered in more detail by proving the convolution theorem and a discrete analogue of the classical theorem of Titchmarsh for the usual continuous convolution.

1. Introduction

A time scale is an arbitrary nonempty closed subset of real numbers. Time scale analysis unifies and extends continuous and discrete analyses; see [1,2]. The Laplace transform on time scales was introduced by Hilger in [3], but in a form that tries to unify the (continuous) Laplace transform and the (discrete) Z-transform. For arbitrary time scales, the Laplace transform was introduced and investigated by Bohner and Peterson in [4] (see also [1, Section 3.10]). It was further developed by the authors in [5,6].

Let $\mathbb{T}$ be a time scale with the forward jump operator σ and the delta differentiation operator Δ. Let $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$ (the so-called graininess of the time scale). A function $p : \mathbb{T} \to \mathbb{C}$ is called *regressive* if

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}.$$

The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{C}$ will be denoted by $\mathcal{R}$. Suppose $p \in \mathcal{R}$ and fix $s \in \mathbb{T}$. Then the initial value problem

$$y^{\Delta}(t) = p(t)y(t), \qquad y(s) = 1 \tag{1.1}$$

has a unique solution on $\mathbb{T}$. This solution is called the *exponential function* and is denoted by $e_p(t, s)$.

Assume that $\sup \mathbb{T} = \infty$ and fix $t_0 \in \mathbb{T}$. Below, we assume that z is a complex constant which is regressive, i.e., $1 + \mu(t)z \neq 0$ for all $t \in \mathbb{T}$. Therefore $e_z(\cdot, t_0)$ is well defined on $\mathbb{T}$. Suppose $x : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is a locally Δ-integrable function, i.e., it is Δ-integrable over each compact subinterval of $[t_0, \infty)_{\mathbb{T}}$. Then the *Laplace transform* of x is defined by

$$\mathcal{L}\{x\}(z) = \int_{t_0}^{\infty} \frac{x(t)}{e_z(\sigma(t), t_0)} \Delta t \quad \text{for } z \in \mathcal{D}\{x\}, \tag{1.2}$$

where $\mathcal{D}\{x\}$ consists of all complex numbers $z \in \mathcal{R}$ for which the improper integral exists.

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The following two concepts were introduced and investigated by the authors in [5]. For a function $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, its *shift* (or *delay*) $\widehat{f}(t, s)$ is defined as the solution of the problem

$$\begin{aligned} &\widehat{f}^{\Delta_t}(t, \sigma(s)) = -\widehat{f}^{\Delta_s}(t, s), \quad t, s \in \mathbb{T}, \ t \geq s \geq t_0, \\ &\widehat{f}(t, t_0) = f(t), \quad t \in \mathbb{T}, \ t \geq t_0. \end{aligned} \tag{1.3}$$

For given functions $f, g : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, their *convolution* $f * g$ is defined by

$$(f * g)(t) = \int_{t_0}^{t} \widehat{f}(t, \sigma(s)) g(s) \Delta s, \quad t \in \mathbb{T}, \ t \geq t_0. \tag{1.4}$$

This paper is organized as follows. In Section 2, we specify and investigate the above concept of Laplace transform for time scales which have graininess that is bounded below by a strictly positive number. These are special cases of so-called isolated time scales. The concept of time scales is actually not needed there and in the remainder of this paper, since all statements and proofs are given directly without referring to this theory. Only this present Section 1 contains time scale concepts and hence shows the origin of this development. However, for a reader to follow the rest of this paper, it is not necessary to be familiar with time scale theory. In Section 3, we present the convolution theorem and a discrete analogue of a classical theorem of Titchmarsh, while Section 4 features a formula for the calculation of the inverse Laplace transform. Finally, in Section 5, we discuss several examples of time scales for which our theory applies, e.g. (see [7,8]) $h\mathbb{Z}$ with $h > 0$, $q^{\mathbb{N}_0}$ with $q > 1$, and $\mathbb{N}_0^p$ with $p \geq 1$.

2. The Laplace transform

Throughout the paper, we let t_n be real numbers for all $n \in \mathbb{N}_0$ such that

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \omega_n := t_{n+1} - t_n > 0 \quad \text{for all } n \in \mathbb{N}_0, \tag{2.1}$$

while we assume in the main results of this paper that

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \omega := \inf_{n\in\mathbb{N}_0} \omega_n > 0, \quad \text{where } \omega_n := t_{n+1} - t_n \text{ for } n \in \mathbb{N}_0 \tag{2.2}$$

holds. Note that, for example, the numbers

$$t_n = hn, \quad n \in \mathbb{N}_0 \quad \text{and} \quad t_n = q^n, \quad n \in \mathbb{N}_0,$$

where $h > 0$ and $q > 1$, respectively, satisfy our assumption (2.2), while the numbers

$$t_n = \sqrt{n}, \quad n \in \mathbb{N}_0 \quad \text{and} \quad t_n = \ln n, \quad n \in \mathbb{N}$$

do not satisfy our assumption (2.2).

Let z be a complex number such that

$$z \neq -\frac{1}{\omega_n} \quad \text{for all } n \in \mathbb{N}_0. \tag{2.3}$$

Then the solution $e_z(t_n, t_m)$ of the (see (1.1)) problem

$$y(t_{n+1}) = (1 + \omega_n z) y(t_n), \qquad y(t_m) = 1, \quad m, n \in \mathbb{N}_0$$

satisfies

$$e_z(t_n, t_m) = \prod_{k=m}^{n-1} (1 + \omega_k z) \quad \text{if } n \geq m \tag{2.4}$$

and

$$e_z(t_n, t_m) = \frac{1}{\prod_{k=n}^{m-1} (1 + \omega_k z)} \quad \text{if } n \leq m,$$

where the products for $m = n$ are understood, as usual, to be 1. Thus, in conformance with (1.2), we make the following definition.

Definition 2.1. Assume (2.1). If $x : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ is a function, then its *Laplace transform* is defined by

$$\widetilde{x}(z) = \mathcal{L}\{x\}(z) = \sum_{n=0}^{\infty} \frac{\omega_n x(t_n)}{\prod_{k=0}^{n} (1 + \omega_k z)} \tag{2.5}$$

for those values $z \in \mathbb{C}$ satisfying (2.3) for which this series converges.