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Computation of scattering resonances for dielectric resonators

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ABSTRACT

In this paper we present a numerical method to compute resonances and resonant modes for 2D electromagnetic scattering at a smooth homogeneous dielectric object in free space. The resonances are found as eigenvalues of a non-linear eigenvalue problem which comes from a formulation as a boundary integral equation and subsequent discretization by a Nystrøm approach, for which the integral kernels are regularized by singularity subtraction. The eigenvalues are computed by a predictor–corrector strategy, which provides good initial guesses for an iterative corrector procedure. The resonances can be computed with very high accuracy due to an exponentially decreasing discretization error.

1. Introduction

Micro-disc resonators have attracted much interest in photonics technology over the past years with various applications (e.g. micro-lasers, optical amplifiers, switches and filters, [1–4] to cite only a few). Several different designs for micro-discs have been studied experimentally and theoretically with emphasis on different topics such as e.g. unidirectional emission or high Q-cavities [5,4,6]. An important tool to judge the performance and optical properties of a given micro-disc are the associated *scattering resonances*.

In this paper we develop an efficient numerical strategy to compute scattering resonances of 2D dielectric objects with high accuracy. These computations can be helpful and valuable for the development and design of novel microstructures, [7].

1.1. Scattering resonances

As model for a micro-disc resonator we consider a homogeneous dielectric rod with given cross-section Ω, cf. Fig. 1(a). Assume that $\Omega \subseteq \mathbb{R}^2$ is an open set in the (ξ_1, ξ_2)-plane and let $\Gamma := \partial\Omega$ be the boundary. Let $x = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3$ and $\varepsilon(x)$ be the electric permittivity at x. Denote by $\boldsymbol{E}(t, x) = (E^1(t, x), E^2(t, x), E^3(t, x))^T$ the electric field and by $\boldsymbol{H}(t, x) = (H^1(t, x), H^2(t, x), H^3(t, x))^T$ the magnetic field. The fields are coupled by Maxwell's equations

$$\varepsilon(x)\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H}$$
$$-\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = \nabla \times \boldsymbol{E},$$

where ε_0, μ_0 are the electric and magnetic constants. Assume TM-polarization, i.e. the magnetic field $\boldsymbol{H}$ is transversal and the electric field $\boldsymbol{E}$ is parallel to the cylinder axis ξ_3. Thus, one has $E^1 = E^2 = 0$ and E^3 depends only on ξ_1 and ξ_2, [3]. Therefore, the problem reduces to the (ξ_1, ξ_2)-plane and we write in the following $x = (\xi_1, \xi_2)^T$ and $E(t, x)$ for $E^3(t, \xi_1, \xi_2, \xi_3)$ with slight abuse of notation. Under these assumptions Maxwell's equations simplify to the wave-equation

$$n^2(x)\, \partial_t^2 E(t, x) = \Delta E(t, x), \qquad (1.1)$$

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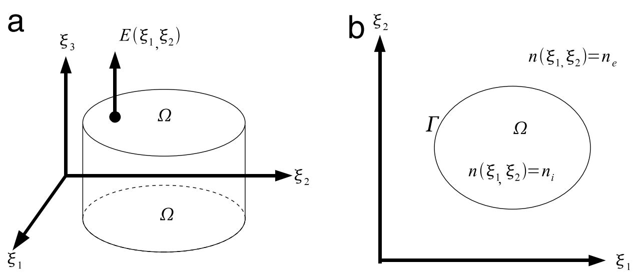

Fig. 1. (a) The micro-disc is modeled by a homogeneous dielectric rod. Assuming TM-polarization the electrical field is parallel to the cylinder axis ξ_3 and satisfies the wave-equation (1.1). (b) The problem reduces to the (ξ_1, ξ_2)-plane. One can distinguish between an interior domain Ω, which is the cross-section of the resonator and an exterior domain $\mathbb{R}^2 \setminus \bar{\Omega}$. The refractive index $n(\xi_1, \xi_2)$ takes only two values, n_i in the interior and n_e in the exterior. It is not defined at the interface $\Gamma := \partial\Omega$.

where the function $n(x) := \sqrt{\varepsilon(x)} \cdot \sqrt{\varepsilon_0\mu_0}$ is the refraction index. Explicitly, we have,

$$n(x) = \begin{cases} n_i : & x \in \bar{\Omega} \\ n_e : & x \notin \Omega \end{cases}$$

with constant index n_i of the surrounding medium and constant index n_e of the rod, cf. Fig. 1(b).

Of special interest for the mathematical analysis and physical applications are time-harmonic solutions of (1.1), which satisfy radiation boundary conditions, [8]. One defines,

Definition 1.1. *Scattering resonances* are solutions to the eigenvalue equation satisfied by time-harmonic solutions $E(x, t) = e^{-ikt}u(x)$ of Eq. (1.1) subject to outgoing radiation conditions, imposed outside the cavity.

This means, we seek non-trivial $u(x; k)$ and k, such that

$$\Delta u(x; k) + k^2 n^2(x)u(x; k) = 0, \tag{1.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad r = |x| \tag{1.3}$$

is satisfied. The condition (1.3) is known as *Sommerfeld's radiation condition*. It guarantees that for large r the field $E(t, x)$ is an outgoing cylindrical wave.

Due to the outgoing radiation conditions the above problem is not self-adjoint and the resonances k are complex valued with $\operatorname{Im} k < 0$. Energy can escape the resonator and is radiated away. This energy loss is measured by the *Q-factor of the resonance*, which is defined by $Q = -\frac{\operatorname{Re} k}{2\operatorname{Im} k}$, [9]. Plugging in the resonance k with $\operatorname{Im} k < 0$ in $E(t, x) = e^{-ikt}u(t, x)$ one sees that $\operatorname{Im} k$ controls the rate of decay of the resonant mode. Therefore, one refers to $\operatorname{Im} k$ as the *lifetime of the resonance k*. For applications, resonators with long-lived resonant modes are most essential, i.e. resonances close to the real axis are required. Because Q is proportional to $\frac{1}{\operatorname{Im} k}$, the imaginary part of the resonance needs to be computed with high accuracy, to avoid large round-off errors in Q. This requires sophisticated computational methods, especially if the effective wave-length $\lambda = \frac{2\pi}{n_i \cdot \operatorname{Re} k}$ of the resonant mode is small compared to the diameter of the resonator.

1.2. Computational methods

To compute resonances and resonant modes of these structures numerically many simulation methods have been applied, including finite element method, finite differences [10,11], scattering matrix approach [12–14] and boundary integral methods [15–17].

In this paper, we focus on a boundary integral discretization to find resonances and modes. For this kind of method Eq. (1.2) is formulated as a boundary integral equation (BIE) using Green's function. This approach has the advantage that the boundary condition at infinity (1.3) has already been embedded in the Green's function, while FEM or FD need to truncate the exterior domain and have to apply artificial boundary condition, for example perfectly matched layers [18]. A second advantage is the reduction to 1D boundary integrals, so that no triangulation of the 2D domain Ω is needed. The price to pay is a full discretization matrix because the integral operators require *global* information in contrast to *local* differential operators. Therefore, it is essential to find discretizations of the integral operators, which admit fast decreasing discretization errors to keep the discretization matrix as small as possible. In our approach we use a singularity subtraction technique together with a specialized quadrature rule for a Nystrøm discretization of the integral operators. This regularization causes an exponentially decreasing discretization error.

With a BIE approach the resonances are found as eigenvalues of a non-linear eigenvalue problem arising from the discretization matrix. To find an eigenvalue Wiersig [16] propose a Newton method and Cho et al. [17] a secant method.