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A new modified secant-like method for solving nonlinear equations

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ABSTRACT

In this paper, we present a new secant-like method for solving nonlinear equations. Analysis of the convergence shows that the asymptotic convergence order of this method is $1+\sqrt{3}$. Some numerical results are given to demonstrate its efficiency.

1. Introduction

Numerical methods for solving nonlinear equations is a popular and important research topic in numerical analysis. In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(x)=0$, where $f: D\subset\mathbb{R}\to\mathbb{R}$ for an open interval D is a scalar function.

Newton's method is an important and basic approach for solving nonlinear equations [1,2], and its formulation is given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This method converges quadratically.

To improve the local order of convergence, a number of modified methods have been studied and reported in the literature (for example, [3–21]). By employing a second-derivative evaluation, we can obtain some well-known third-order methods, such as Chebyshev's method, Halley's method and the super-Halley method [3,4]. In order to replace the second derivative, an evaluation of the function or first derivative is added, and then many third-order and higher-order methods are obtained (for example, [5–19]).

However, in many other cases, it is expensive to compute the first derivative, and the above methods are still restricted in practical applications. The well-known secant method is given by

$$x_{n+1}=x_n-\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}f(x_n). \tag{2}$$

This method can be derived by finding the root of the linear polynomial function

$$L_1(x)=f(x_n)+\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}(x-x_n). \tag{3}$$

This method does not require any derivative, but its order is only 1.618.

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In order to improve this method, Zhang et al. [22] consider

$$L_2(x) = f(x_n) + \frac{f(y_n) - f(x_n)}{y_n - x_n}(x - x_n), \tag{4}$$

and then find x_{n+1} such that $L_2(x_{n+1}) = 0$. From this, they obtain

$$x_{n+1} = x_n - \frac{x_n - y_n}{f(x_n) - f(y_n)} f(x_n), \tag{5}$$

where y_{n+1} is defined by

$$y_{n+1} = x_{n+1} - \frac{x_n - y_n}{f(x_n) - f(y_n)} f(x_{n+1}). \tag{6}$$

This method is also a secant-like method, and the order is improved to 2.618.

In this paper, we attempt to improve the order of the method proposed in [22] by using previous information, and then we present a new iterative method for solving nonlinear equations. Analysis of the convergence shows that the asymptotic convergence order of this method is $1 + \sqrt{3}$. The practical utility is demonstrated by numerical results.

2. Notation and basic results

Let $f(x)$ be a real function with a simple root x^* and let $\{x_n\}_{n\in N}$ be a sequence of real numbers that converges to x^*. We say that the *order of convergence* is q if there exists a $q \in \mathbb{R}^+$ such that

$$\lim_{n\to+\infty} \frac{x_{n+1} - x^*}{(x_n - x^*)^q} = C \neq 0, \infty.$$

Let $e_n = x_n - x^*$ be the nth iterate error. We call

$$e_{n+1} = Ce_n^q + \cdots \tag{7}$$

the *error equation*, in which the higher-order terms are neglected. If we can obtain the error equation for the method, then the value of q is its order of convergence.

3. The method and its convergence

Here, in order to construct our method, we use the following the second-order polynomial function:

$$P(x) = f(x_n) + \upsilon_n^{-1}(x - x_n) + \frac{\left(\upsilon_{n-1}^{-1} - \upsilon_n^{-1}\right)(x - x_n)(x - y_n)}{\alpha_1 x_{n-1} + \alpha_2 y_{n-1} + (2 - \alpha_1 - \alpha_2) z_{n-1} - \beta_1 x_n - \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, and y_n, z_n and υ_n are defined by $y_n = x_n - \nu_{n-1} f(x_n)$, $\nu_n = (y_n - x_n)/(f(y_n) - f(x_n))$ and $z_n = x_n - \nu_n f(x_n)$, respectively.

It is easy to obtain that

$$\left(\upsilon_{n-1}^{-1} - \upsilon_n^{-1}\right)(y_n - x_n)(z_n - y_n) = \upsilon_n^{-1}(y_n - z_n)^2.$$

In order to eliminate the nonlinearity, we replace x_{n+1} in the terms $(x_{n+1} - x_n)$ and $(x_{n+1} - y_n)$ of $P(x_{n+1})$ with y_n and z_n, respectively, and then finding its solution, we use the following new method:

$$\begin{cases} y_n = x_n - \nu_{n-1} f(x_n), \\ \nu_n = (y_n - x_n)/(f(y_n) - f(x_n)), \\ z_n = x_n - \nu_n f(x_n), \\ x_{n+1} = z_n - \dfrac{(y_n - z_n)^2}{\alpha_1 x_{n-1} + \alpha_2 y_{n-1} + (2 - \alpha_1 - \alpha_2) z_{n-1} - \beta_1 x_n - \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n}, \end{cases} \tag{8}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.

The method defined by (8) can be viewed as an iterative method with three substeps. The first two substeps are a variant of the method proposed in [22]. The third substep is an acceleration by using the values computed previously.

At the beginning of the process, the value of ν_{-1} needs to be given by some approaches. One choice of ν_{-1} is given by

$$\nu_{-1} = \varepsilon,$$

while another choice is

$$\nu_{-1} = \frac{\varepsilon f(x_0)}{f(x_0 + \varepsilon f(x_0)) - f(x_0)}.$$