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The point of coincidence and common fixed point for a pair of mappings in cone metric spaces

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ABSTRACT

In this paper the existence of a point of coincidence and a common fixed point for two weakly compatible maps on a cone metric space has been established. The two mappings are assumed to satisfy certain weak inequalities. Supporting examples are also given.

1. Introduction

Recently, Huang and Zhang [1] introduced the concept of a cone metric space where every pair of elements is assigned to an element of a Banach space equipped with a cone which induces a natural partial order. They proved some fixed point theorems for such spaces in the same work. After that, fixed point results for cone metric spaces were studied by many other authors. References [2–9] are some works in this line of research.

The weak contraction principle was first given by Alber et al. for Hilbert spaces [10] and subsequently extended to metric spaces by Rhoades [11]. After that, fixed point problems involving weak contractions and mappings satisfying weak contraction type inequalities were considered in several works like [12–17]. In particular, in cone metric spaces the weak contraction principle was extended by the present authors [18].

In the present paper we prove three results on points of coincidence and common fixed points for two weakly compatible mappings in cone metric spaces by using a control function, where these mappings are assumed to satisfy certain weak inequalities. It may be mentioned that some fixed point results for weakly compatible maps in cone metric spaces have been deduced by Abbas and Jungck [2]. The use of a control function in fixed point theory was initiated by Khan et al. [19]; they called it an altering distance function. This function has been used in obtaining fixed point results for metric spaces [20–22] and probabilistic metric spaces [23,24]. It has also been used in multivalued and fuzzy fixed point problems [25].

2. Mathematical preliminaries

Definition 2.1 (*[1]*)**.** Let E always be a real Banach space and P a subset of E. P is called a cone if and only if:

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(i) P is nonempty, closed, and $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Longrightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subset E$, a partial ordering $\leq$ with respect to P is naturally defined by $x \leq y$ if and only if $y - x \in P$, for $x, y \in E$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is said to be normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above statement is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent; that is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose that E is a real Banach space with cone P in E with $\operatorname{int} P \neq \emptyset$ and $\leq$ is the partial ordering with respect to P.

Definition 2.2. Let $\psi : P \longrightarrow P$ be a function.

(i) We say ψ is strongly monotone increasing if for $x, y \in P, x \leq y \Longleftrightarrow \psi(x) \leq \psi(y)$.
(ii) ψ is said to be continuous at $x_0 \in P$ if for any sequence $\{x_n\}$ in P, $x_n \longrightarrow x_0 \Longrightarrow \psi(x_n) \longrightarrow \psi(x_0)$.

Definition 2.3 (*[1]*)**.** Let X be a nonempty set. Let the mapping $d : X \times X \longrightarrow E$ satisfy:

(i) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.4 (*[1]*)**.** Let (X, d) be a cone metric space and $\{x_n\}$ a sequence in X.

(i) $\{x_n\}$ converges to $x \in X$ if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbf{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$. We denote this by $\lim_n x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow \infty$.
(ii) If for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbf{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that if P is a normal cone, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$ [1].

Definition 2.5 (*[26]*)**.** Let f and g be self-maps of a set X (i.e., $f, g : X \longrightarrow X$). If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self-maps f and g are said to be weakly compatible if they commute at their coincidence point; that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

The results noted in the following lemmas will be used in deriving all theorems. The proofs of these lemmas give references to the respective works in which they appear.

Lemma 2.1 (*[2]*)**.** *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Lemma 2.2. *Let E be a real Banach space with cone P in E. Then:*

(i) *if $a \leq b$ and $b \ll c$, then $a \ll c$* [5],
(ii) *if $a \ll b$ and $b \ll c$, then $a \ll c$* [5],
(iii) *if $0 \leq x \leq y$ and $a \geq 0$, where a is real number, then $0 \leq ax \leq ay$* [5],
(iv) *if $0 \leq x_n \leq y_n$, for $n \in \mathbf{N}$ and $\lim_n x_n = x$, $\lim_n y_n = y$, then $0 \leq x \leq y$* [5],
(v) *P is normal if and only if $x_n \leq y_n \leq z_n$ and $\lim_n x_n = \lim_n z_n = x$ imply $\lim_n y_n = x$* [27].

A control function ψ was introduced in the following result in [19].