Computers and Mathematics with Applications 60 (2010) 1723–1738

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An iterative Lagrange method for the regularization of discrete ill-posed inverse problems

G. Landi *, E. Loli Piccolomini

Department of Mathematics, University of Bologna, 40138, Italy

ARTICLE INFO

Article history:
Received 1 February 2010
Received in revised form 2 July 2010
Accepted 6 July 2010

Keywords:
Ill-posed problems
Inverse problems
Regularization methods
Lagrange method
Discrepancy principle

ABSTRACT

In this paper, an iterative method is presented for the computation of regularized solutions of discrete ill-posed problems. In the proposed method, the regularization problem is formulated as an equality constrained minimization problem and an iterative Lagrange method is used for its solution. The Lagrange iteration is terminated according to the discrepancy principle. The relationship between the proposed approach and classical Tikhonov regularization is discussed. Results of numerical experiments are presented to illustrate the effectiveness and usefulness of the proposed method.

1. Introduction

In many applications, the discretization of continuous ill-posed inverse problems, such as Fredholm integral equations of the first kind, results in discrete ill-posed problems of the form

$$Ax = b \tag{1}$$

where $A \in R^{n\times n}$ is a nonsingular ill-conditioned matrix.

The vector b is contaminated by noise and measurement errors, i.e.,

$$b = \widetilde{b} + e$$

where the noise-free vector $\widetilde{b}$ is unavailable and e represents perturbations and measurement errors. The error norm

$$\sigma := \|e\|$$

is assumed to be explicitly known. (Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm.) Our aim is to compute a good approximation of the solution $\widetilde{x}$ of the noise-free linear system (1) even in the presence of noisy data b.

Due to the ill-conditioning of A and the noise contained in b, the direct solution $\widehat{x}$ of (1) is a poor approximation of $\widetilde{x}$ and a regularization method is necessary in order to determine a useful approximation of $\widetilde{x}$. (For thorough discussions on regularization, please refer to [1–4] and the references therein.) Tikhonov regularization [5] is one of the most popular regularization techniques. In this method, the original problem (1) is replaced by the least-squares problem

$$\text{minimize} \quad \frac{1}{2}\|Ax-b\|^2 + \frac{\mu}{2}\|L(x-x_{\text{guess}})\|^2 \tag{2}$$

* Corresponding author. Tel.: +39 0512094480; fax: +39 0512094490.
E-mail addresses: landig@dm.unibo.it (G. Landi), piccolom@dm.unibo.it (E. Loli Piccolomini).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2010.07.003

where μ is a positive regularization parameter, x_{guess} is an *a priori* estimate of $\widetilde{x}$ and $L \in \mathbb{R}^{p\times n}$, $p \le n$, is a full-rank regularizing matrix containing prior information about the smoothness of the exact solution. For example, L can be the identity matrix or the discrete approximation of the first-order or second-order derivative operators. In practice, x_{guess} is usually set to zero because no information on the solution is available.

The effectiveness of Tikhonov regularization strongly depends on a suitable choice of the regularization parameter μ which determines the amount of regularization. In fact, if μ is too small, noise will dominate the solution, while if μ is too large, the resulting solution will be too smooth. When the error norm σ is explicitly known, a popular and useful parameter choice strategy is the *discrepancy principle* [1,6–8]. This principle states that the optimal regularization parameter μ gives a regularized solution whose residual norm is equal to $\rho\sigma$ for some parameter $\rho > 1$.

Tikhonov regularization (2) can be equivalently reformulated as one of the following constrained optimization problems:

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\|Lx\|^2 \\ &\text{subject to} \quad \frac{1}{2}\|Ax-b\|^2 = \frac{\sigma^2}{2} \end{aligned} \tag{3}$$

or

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\|Ax-b\|^2 \\ &\text{subject to} \quad \frac{1}{2}\|Lx\|^2 = \frac{1}{2}\|L\widetilde{x}\|^2. \end{aligned} \tag{4}$$

It can be proved ([9] and the references therein) that problems (2)–(4) are equivalent when $x_{\text{guess}} = 0$ provided that μ is the exact Lagrange multiplier of (4) or the inverse of the exact Lagrange multiplier of (3). The advantage of the constrained formulations (3) and (4) over the unconstrained one (2) consists in not requiring any prior estimate of the regularization parameter but rather requiring an estimate of the error norm or the exact solution norm. In many applications, these estimates are obtainable.

Both the noise constrained optimization problems (3) and (4) have been considered in the literature for the regularization of discrete ill-posed problems. In [9], efficient algorithms based on a bidiagonalization of A are presented for (3) and (4). In [10], the problem (4) is considered as a special case of the trust region subproblem from optimization and the LSTRS [11] method for large-scale trust region subproblems is applied to (4). In [12,13], methods based on Gauss quadrature are proposed for problem (4). In [14], a modular approach, involving the solution of a sequence of unconstrained problems while adjusting the regularization parameter, is presented for the solution of (3). In [15], problem (3) is considered and a Lagrange method [16,17] is proposed for its solution. The Lagrange method presented consists in solving the first-order optimality conditions of (3) with respect to both the variable x and the Lagrange multiplier by means of an inexact Newton-like method. The search direction is computed by approximately solving the Newton system of the first-order conditions with the minimal residual (MINRES) method of Paige and Saunders [18] and the step length is determined by a constrained line search involving a penalized merit function. The penalized merit function is introduced to penalize the constraint violations. The constrained line search modifies the step length not only in order to ensure a sufficient decrease of the merit function but also to ensure the positivity of the Lagrange multiplier and the nonsingularity of the Newton system coefficient matrix.

In this work, we propose a regularization strategy based on replacing the original problem (1) with the constrained minimization problem

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\|Lx\|^2 \\ &\text{subject to} \quad \frac{1}{2}\|Ax-b\|^2 = 0 \end{aligned} \tag{5}$$

which is different from problems (3) and (4) proposed in the literature. A useful approximation to the true solution $\widetilde{x}$ is determined "by iteration with early stopping". An iterative Lagrange method is applied to (5) and a regularized solution of (1) is obtained by stopping its iteration according to the discrepancy principle. That is, the Lagrange iterations are terminated as soon as an iterate x_k has been determined whose residual norm $\|Ax_k - b\|$ is less than $\rho\sigma$. We will refer to the proposed regularization technique as the *truncated Lagrange (TL) method*.

The main contribution of this work is to analyze the relationship between the proposed regularization strategy, based on solving (5) with an iterative procedure, and classical Tikhonov regularization (2). In particular, in this work, we show that the solution computed by the TL method solves a Tikhonov regularization problem of the form (2) where the regularization parameter μ satisfies the discrepancy principle and x_{guess} is a suitable estimate of the exact solution $\widetilde{x}$.

We believe that the main advantage of the proposed approach over classical Tikhonov regularization (2) is that it removes the indefiniteness of μ and x_{guess} since both of these values are estimated during the iterations of the TL method. Only the knowledge of the noise norm σ is necessary for use in the discrepancy principle. In practical applications, a useful *a priori* estimate of the regularization parameter is often unavailable and, in spite of the numerous methods proposed in the