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Uniqueness and periodicity of meromorphic functions concerning the difference operator

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ABSTRACT

In this paper, we investigate the uniqueness problems of difference polynomials of meromorphic functions that share a value or a fixed point. We also obtain several results concerning the shifts of meromorphic functions and the sufficient conditions for periodicity which improve some recent results in Heittokangas et al. (2009) [10] and Liu (2009) [11].

1. Introduction

A function $f(z)$ is called meromorphic if it is analytic in the complex plane $\mathbb{C}$ except at possible isolated poles. If no poles occur, then $f(z)$ reduces to an entire function. In what follows, we assume that the reader is familiar with the basic results and notation of Nevanlinna theory [1,2].

We say that two meromorphic functions $f(z)$ and $g(z)$ share a small function $a(z)$ IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities). We denote by $N_p\left(r,\frac{1}{f-a}\right)$ the counting function of the zeros of $f-a$, where an m-fold zero is counted m times if $m\le p$ and p times if $m>p$. We use $\sigma(f)$ to denote the order of a meromorphic function $f(z)$. Denote by $\mathbb{S}(f)$ the family of all meromorphic functions $a(z)$ that satisfy $T(r,a)=o(T(r,f))$, where $r\to\infty$ outside a possible exceptional set of finite logarithmic measure. For convenience, we set $\hat{\mathbb{S}}(f):=\mathbb{S}(f)\cup\{\infty\}$.

Let $f(z)$ be a transcendental meromorphic function, n be a positive integer. Many authors have investigated the value distributions of f^nf'. In 1959, Hayman [3] proved that f^nf' takes every non-zero complex value infinitely often if $n\ge 3$. The case $n=2$ was settled by Mues [4] in 1979. Bergweiler and Eremenko [5] showed that $ff'-1$ has infinitely many zeros.

Laine and Yang [6, Theorem 2] investigated, corresponding to the above results, the value distribution of difference products of entire functions, and obtained the following result.

Theorem A. *Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n\ge 2$, $f^n(z)f(z+c)$ assumes every non-zero value $a\in\mathbb{C}$ infinitely often.*

Recently, Liu and Yang [7, Theorem 1.2] improved Theorem A and obtained the next result.

Theorem B. *Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n\ge 2$, $f^n(z)f(z+c)-p(z)$ has infinitely many zeros, where $p(z)$ is a non-zero polynomial.*

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As regards the uniqueness problems for entire functions, Fang and Hua [8] and also Yang and Hua [9] obtained some results. One of them can be stated as follows.

Theorem C. *Let f and g be non-constant entire functions, and let $n \geq 6$ be an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In this paper, we consider the uniqueness problems concerning the difference polynomials of entire functions, and obtain the following results.

Theorem 1.1. *Let f and g be transcendental entire functions of finite order, and c be a non-zero complex constant; let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share z CM, then $f = t_1 g$ for a constant t_1 that satisfies $t_1^{n+1} = 1$.*

Theorem 1.2. *Let f and g be transcendental entire functions of finite order, and c be a non-zero complex constant; let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share 1 CM, then $fg = t_2$ or $f = t_3 g$ for some constants t_2 and t_3 that satisfy $t_2^{n+1} = 1$ and $t_3^{n+1} = 1$.*

Remark. Let $f(z) = e^z$, $g(z) = e^{-z}$. It is easy to verify that $f^n f(z+c)$ and $g^n g(z+c)$ share 1 CM for any positive integer n and constant c. This implies that the former case of Theorem 1.2 may occur.

Recently, Heittokangas et al. [10] investigated the periodicity of meromorphic functions that share three small periodic functions with their shifts and obtained many results. Some improvements can be found in [11]. Here, we just recall the following two results.

Theorem D *([10, Theorem 7]). Let f be a transcendental meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \hat{\mathbb{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_3 CM, and if*

$$\limsup_{r\to\infty} \frac{N_2\left(r, \frac{1}{f-a_1}\right) + N_2\left(r, \frac{1}{f-a_2}\right)}{T(r,f)} < \frac{1}{2},$$

then $f(z) = f(z+c)$ or $f(z) = f(z+2c)$ for all $z \in \mathbb{C}$.

Theorem E *([10, Theorem 8]). Let f be a transcendental meromorphic function of finite order, $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \hat{\mathbb{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_3 IM, and if*

$$\overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) = S(r,f),$$

then $f(z) = f(z+c)$ or $f(z) = f(z+2c)$ for all $z \in \mathbb{C}$.

In this paper, we will continue to improve the conditions of the above two theorems and obtain the following results.

Theorem 1.3. *Let f be a transcendental meromorphic function of finite order, $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \hat{\mathbb{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_3 IM, and if one of the following conditions is satisfied:*

$$\text{(i) } \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) < \frac{1}{8}T(r,f), \tag{1.1}$$

$$\text{(ii) } N_2\left(r, \frac{1}{f-a_1}\right) + N_2\left(r, \frac{1}{f-a_2}\right) < \frac{1}{6}T(r,f), \tag{1.2}$$

then $f(z) = f(z+c)$ or $f(z) = f(z+2c)$ for all $z \in \mathbb{C}$.

Corollary 1. *Let f be a transcendental meromorphic function of finite order, $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \hat{\mathbb{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_3 IM, and if*

$$N\left(r, \frac{1}{f-a_1}\right) + N\left(r, \frac{1}{f-a_2}\right) < \frac{1}{6}T(r,f), \tag{1.3}$$

then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

In [10], Heittokangas et al. also obtained an analogue of the Brück conjecture [12], which can be stated as follows.

Theorem F. *Let f be a meromorphic function of order $\sigma(f) < 2$, and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share the values $a \in \mathbb{C}$ and ∞ CM, then*

$$\frac{f(z+c)-a}{f(z)-a} = \tau$$

for a non-zero constant τ.