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Parallel solution of the modified porous medium equation

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A B S T R A C T

The present study deals with the numerical solution of the modified porous medium equation when the solution is subject to some constraints. First of all, we use a change of variables, which leads to an evolutive problem where the nonlinear part is constituted by the derivative with respect to the time, of a diagonal increasing operator. Then, for the numerical solution we consider an implicit time marching scheme, which leads to the solution of a sequence of stationary problems. In fact each stationary problem is equivalent to a constrained minimization problem, and for Dirichlet and Dirichlet–Neumann boundary conditions, we show the existence and the uniqueness of the solution of our stationary constrained minimization problem. Moreover, classically, the solution of each stationary constrained problem can be characterized by the solution of a multivalued one. The spatial discretization of the previous problem leads to the solution of large scale algebraic multivalued systems. Then we analyze in a unified approach the convergence of the sequential and parallel relaxation projected methods. Finally we present the results of numerical experiments.

1. Introduction

The fundamental principle of the theory of hydrodynamic limits is to study the evolution of the transition from microscopic to macroscopic dynamics of particle systems. At the microscopic level the evolution of the particles is modeled on the microscopic space $\mathbb{T}_N^d$ by the random dynamic, where $\mathbb{T}_N^d = \{0, \dots, N-1\}^d$ is the discrete d-dimensional torus of the size N.

The macroscopic hydrodynamic behavior of the system is described on the macroscopic space $\Omega = ([0, 1[)^d$, the d-dimensional torus, and is obtained after the space scaling change of the microscopic system where the parameter of the spatial scaling change is N^{-1}. At the initial time, the particles are distributed according to the density $u_0 : \Omega \longrightarrow \mathbb{R}_+$; at the time t the system is described by the density of particles $u_t : \Omega \longrightarrow \mathbb{R}_+$ which is the solution of the partial differential equation of the parabolic type under the previous scaling change called hydrodynamic equation, that governs the macroscopic evolution of a fluid or a gas evolving in a volume from a microscopic dynamics at random due to the large number of particles.

In [1] Gonçalves, Landim and Toninelli established the hydrodynamic limit for some particle systems with degenerate rates under exclusive constraints such that the macroscopic density profile evolves under the diffusive time scaling according to a porous medium equation for initial profiles. Recently Sasada in [2] established, for conservative particle

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systems on the d-dimensional discrete torus $\mathbb{T}_N^d$ with degenerate jump rates without restrictions on the total number of particles per site, the same result according to a modified porous medium equation (*MPME*). For a large survey on hydrodynamic limits, one may refer to Spohn [3] or Kipnis and Landim [4].

In the sequel we will denote by $\alpha, \beta, \gamma, \ldots$ the elements of $\mathbb{T}_N^d$, $x, y, \ldots$ the elements of Ω.

1.1. Motivation of the study

Let us consider a particle system which evolves according to a continuous time Markov process $(\xi_t)_{t\geq 0}$ with state space $\chi_N^d = \mathbb{N}^{\mathbb{T}_N^d}$, where $\mathbb{T}_N^d = \{0, \ldots, N-1\}^d$ is the d-dimensional torus (see [5]). Let $\xi(\alpha)$ denote a configuration in χ_N^d such that $\xi(\alpha) = k$ if there are k particles at α. The process is defined through the function $g(k) = \frac{k}{\theta+k-1} : \mathbb{N} \longrightarrow \mathbb{R}_+$ vanishing at 0, as described in [2] (see example 3.1), as follows. The moves of particles, which are subject to a speed represented by g, during evolution of the system among nearest neighbors α and β occurred with rate $\frac{c(\alpha,\beta,\xi)}{\theta+\xi(\alpha)-1}$, where $c(\alpha, \beta, \xi)$ denotes exchange rate. Namely, one of the particles at α jumps to β with rate $\frac{kc(\alpha,\beta,\xi)}{\theta+k-1}$ if there are k particles at a site α.

The dynamics of particles is defined by means of an infinitesimal generator acting on cylinder function $f : \chi_N^d \longrightarrow \mathbb{R}$ as

$$(\mathcal{L}_N f)(\xi) = \sum_{\alpha,\beta\in\mathbb{T}_N^d, |\alpha-\beta|=1} \frac{\xi(\alpha).c(\alpha,\beta,\xi)}{\theta+\xi+(\alpha)-1}\left(f\left(\xi^{\alpha,\beta}\right) - f(\xi)\right),$$

where $\theta > 0$ and $|\alpha - \beta| = \sum_{1\leq i\leq d} |\alpha_i - \beta_i|$ is the sum norm in $\mathbb{R}^d$ and

$$\xi^{\alpha,\beta}(\gamma) = \begin{cases} \xi(\gamma) & \text{if } \gamma \neq \alpha, \beta \\ \xi(\alpha) - 1 & \text{if } \gamma = \alpha \\ \xi(\beta) + 1 & \text{if } \gamma = \beta. \end{cases}$$

In [2] Sasada uses the relative entropy method to establish the hydrodynamic limit, which consists to identifying the equations that give a description to the macroscopic scale phenomena considered, the data of a particle system corresponding to the description at the microscopic scale and a kinetic equation which describes the system at the macroscopic scale. In this model the equation in question is the modified porous medium equation given on the edge Ω by

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta\left(\left(\dfrac{u(t,x)}{u(t,x)+\theta}\right)^m\right), & (t,x) \in [0,T] \times \Omega \\ u(0,.) = \phi(.) \end{cases} \tag{1}$$

where $m \in \mathbb{N}\backslash\{0, 1\}$ is the number of authorized moving of particles and ϕ is an initial profile on $\mathbb{R}_+^\Omega$ of class $C^{2+\varepsilon}(\Omega)$, $\varepsilon > 0$ satisfying the bounded condition ($\delta_0 \leq \phi(x) \leq \delta_1$), where δ_0, δ_1 are nonnegative constants.

By the theorem A 2.4.1 of [6], the equation (1) admits a solution $u(t, x)$ which is of class $C^{1+\varepsilon,2+\varepsilon}([0, T] \times \Omega)$ and

$$\delta_0 \leq \inf_{t,x} u(t,x) \leq \sup_{t,x} u(t,x) \leq \delta_1. \tag{2}$$

1.2. Presentation of the study

For Dirichlet and Dirichlet–Neumann boundary conditions, the goal of the present study is on one hand to prove the existence and the uniqueness of the solution of the stationary continuous problem derived after appropriate temporal discretization from the model problem (1) and on the other hand to solve by various general numerical relaxation methods this previous system of equations.

We have to solve a strongly nonlinear problem where the solution u is submitted to some constraints. After an appropriate change of variables we have to solve a strongly nonlinear boundary value problem, where the operator arising after transformation, is now constituted by a Laplacian perturbed by the derivative with respect to the time of a diagonal nonlinear increasing operator; moreover the new resulting changed solution, now denoted by v, is also subject to some constraints.

Since the previous resulting boundary value problem is always time dependent, we consider a temporal discretization by an implicit time marching scheme; thus, at each time step, we have to solve a sequence of stationary nonlinear and constrained problems.

Since the linear part constituted by a Laplacian is self adjoint we can formulate the transformed problem like a constrained minimization problem on a closed convex set. Then, we can verify by a direct way that the constrained stationary minimization problem has a unique solution for classical boundary conditions, i.e. Dirichlet boundary condition and Dirichlet–Neumann boundary condition; nevertheless note that we can also obtain the same result by applying directly a classical result of [7]. Note that such result is not valid when the model problem is equipped with Neumann boundary condition or Fourier (or Robin) boundary conditions.

Nevertheless, the previous formulation of the constrained optimization problem is not easy to use when, after spatial discretization by classical finite difference schemes, we consider the numerical solution of the discrete model problem, particularly for the analysis of the behavior of the iterative algorithm used to solve the model problem.