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On a homogeneous recurrence relation for the determinants of general pentadiagonal Toeplitz matrices

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ABSTRACT

Pentadiagonal Toeplitz matrices frequently arise in many application areas and have been attracted much attention in recent years. In this paper, we present a numerical algorithm of $O(\log n)$ for computing the determinants of general pentadiagonal Toeplitz matrices without imposing any restrictive conditions. In addition, we investigate some special pentadiagonal Toeplitz determinants and their relations to well-known number sequences, and give a few identity formulas for the ordinary Fibonacci sequences and the generalized k-Fibonacci sequences.

1. Introduction

In general, a pentadiagonal Toeplitz (PT) matrix is defined as having zeros everywhere except in its five principal diagonals, with each principal diagonal having the same element in all positions. For notational purpose, we often choose to write this matrix in the single line form, i.e., $P = [e, d, a, b, c]$.

From theoretical point of view, a recursive relation of one of the computational formulas for PT determinants is used in the inverse problem of constructing symmetric PT matrices from three largest eigenvalues [1,2]. Moreover, it is known that there are close relations between determinants of special PT matrices and some well-known number sequences such as Pell sequence, Jacobsthal sequence, Fibonacci sequence and k-Fibonacci sequence, see [3,4]. On the other hand, from practical point of view, PT matrices frequently arise from boundary value problems (BVP) involving fourth order derivatives and fast computational formulas for the determinants are required to test efficiently the existence of unique solutions of the PDEs, see, e.g., [5–7].

Usually, the determinant of an n-by-n PT matrix P_n can be computed by the Leibniz formula

$$\det(P_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i},$$

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where $\mathrm{sgn}(\cdot)$ is the sign function of permutation in the permutation group S_n. Also, the determinant of the matrix can be given by

$$\det(P_n) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij},$$

where M_{ij} is the (i,j)-minor of P_n. This is the well-known Laplace formula (also called cofactor expansion) in linear algebra. However, methods of implementing an algorithm to compute the determinant by using the above formulas are extremely inefficient for large matrices, since the number of required operations grows very quickly. For example, Leibniz formula requires to calculate $n!$ products.

Recently, some authors have devised numerical (or symbolic) algorithms for PT determinants, see [8–12]. And, the algorithms given in these references require $11n-17, 9n+3, 8n+10, 56\lfloor\frac{n-4}{k}\rfloor+30k+O(\log n), O(\frac{3}{2}k^2\log_2\frac{n}{k}+s^3)$ operations, respectively. For related works such as computing the inverse, determinants and eigenvalues of general pentadiagonal matrices, and solving the pentadiagonal linear systems, see e.g., [13–24] and references therein. The motivation of this paper is to develop a new computational algorithm (with less complexity) for computing the nth order PT determinants.

The remaining part of this paper is organized as follows: In Section 2, we propose an algorithm for the general PT determinants and discuss its computational costs. Illustrative examples and remarks are also given. In addition, we present a few identity formulas for the ordinary Fibonacci sequence and the generalized k-Fibonacci sequence. Some concluding remarks are offered in Section 3.

2. Main results

In this section, we will develop a numerical algorithm for computing the PT determinants without imposing any restrictive conditions. Below, we introduce two auxiliary matrices for later use.

$$Q_i := \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] \in \mathbb{R}^{i\times i}, \qquad R_i := \left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline R_{21} & R_{22} \end{array}\right] \in \mathbb{R}^{i\times i}, \quad i \geq 4,$$

where $Q_{11} = d \in \mathbb{R}$, $Q_{12} = [b, c, 0, \ldots, 0] \in \mathbb{R}^{1\times(i-1)}$, $Q_{21} = [e, 0, \ldots, 0]^T \in \mathbb{R}^{(i-1)\times 1}$, $Q_{22} = P_{i-1} \in \mathbb{R}^{(i-1)\times(i-1)}$, $R_{11} = \begin{bmatrix} d & a \\ e & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $R_{12} = \begin{bmatrix} c & 0 & 0 & \cdots & 0 \\ b & c & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{2\times(i-2)}$, $R_{21} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ e & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{R}^{(i-2)\times 2}$, and $R_{22} = P_{i-2} \in \mathbb{R}^{(i-2)\times(i-2)}$. Here, the superscript symbol T corresponds to the transpose operation, and P_i denotes the i-by-i PT matrix.

2.1. An approach based on homogeneous recurrence relations

By applying Laplace expansion to matrix P_i (Q_i and R_i) on the first row, it yields

$$\det(P_i) = a \cdot \det(P_{i-1}) - b \cdot \det(Q_{i-1}) + c \cdot \det(R_{i-1}), \tag{2.1}$$

$$\det(Q_i) = d \cdot \det(P_{i-1}) - be \cdot \det(P_{i-2}) + ce \cdot \det(Q_{i-2}), \tag{2.2}$$

$$\det(R_i) = d \cdot \det(Q_{i-1}) - ae \cdot \det(P_{i-2}) + ce^2 \cdot \det(P_{i-3}). \tag{2.3}$$

Together with (2.1) and (2.3), we can deduce that

$$\det(P_i) - a \cdot \det(P_{i-1}) + ace \cdot \det(P_{i-3}) - c^2e^2 \cdot \det(P_{i-4}) + b \cdot \det(Q_{i-1}) - cd \cdot \det(Q_{i-2}) = 0. \tag{2.4}$$

Meanwhile, it follows from (2.2) that

$$\begin{aligned} b \cdot \det(Q_{i-1}) - cd \cdot \det(Q_{i-2}) = {} & bd \cdot \det(P_{i-2}) - (b^2e + cd^2) \cdot \det(P_{i-3}) + bcde \cdot \det(P_{i-4}) \\ & + ce(b \cdot \det(Q_{i-3}) - cd \cdot \det(Q_{i-4})). \end{aligned} \tag{2.5}$$

Together with (2.4) and (2.5), we can obtain a seven-term homogeneous recurrence relation as follows:

$$\begin{aligned} \det(P_i) = {} & a \cdot \det(P_{i-1}) - (bd - ce) \cdot \det(P_{i-2}) - (2ace - b^2e - cd^2) \cdot \det(P_{i-3}) \\ & - ce(bd - ce) \cdot \det(P_{i-4}) + ac^2e^2 \cdot \det(P_{i-5}) - c^3e^3 \cdot \det(P_{i-6}). \end{aligned} \tag{2.6}$$

In addition, the associated characteristic equation of (2.6) is defined by

$$\lambda^6 - a\lambda^5 + (bd - ce)\lambda^4 + (2ace - b^2e - cd^2)\lambda^3 + ce(bd - ce)\lambda^2 - ac^2e^2\lambda + c^3e^3 = 0. \tag{2.7}$$

Theorem 2.1. *Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Suppose the characteristic equation*

$$\lambda^k - c_1\lambda^{k-1} - \cdots - c_k = 0,$$