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Ordered cone metric spaces and fixed point results

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ABSTRACT

In this paper, we introduce a partial order on a cone metric space and prove a Caristi-type theorem. Furthermore, we prove fixed point theorems for single-valued nondecreasing and weakly increasing mappings, and multi-valued mappings on an ordered cone metric space.

1. Introduction

Non-convex analysis, especially ordered normed spaces, normal cones and Topical functions [1–7], has several applications in optimization theory. In these cases an order is introduced by using vector space cones. Huang and Zang [5] used this approach, and they replaced the real numbers by ordering Banach space and defined a cone metric space. Also, they proved some fixed point theorems of contractive mappings on this new setting.

After the definition of the concept of cone metric space in [5], fixed point theory on these spaces has been developing (see, e.g., [1,8–14,6,15–24,7,25–29]). Generally, this theory on cone metric space is used for contractive-type or contractive-type mappings (see the related references [1–29]). On the other hand, fixed point theory on partially ordered sets has also been developing recently [10,11,30–32].

In this paper, we introduce a partial order on a cone metric space and prove a Caristi-type theorem. Furthermore, we prove fixed point theorems for single-valued nondecreasing and weakly increasing mappings, and multi-valued mappings on an ordered cone metric space.

We recall the definition of cone metric spaces and some of their properties [5]. Let E be a real Banach space and P be a subset of E. By θ we denote the zero element of E and by $\operatorname{Int} P$ the interior of P. The subset P is called a cone if and only if

(i) P is closed, nonempty and $P \neq \{\theta\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Longrightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Longrightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, and we shall write $x \ll y$ if $y - x \in \operatorname{Int} P$.

The cone P is called normal if there is a number $M > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies that $\|x\| \leq M \|y\|$.

The least positive number satisfying the above is called the normal constant of P.

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The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It has been proved in Lemma 1.1 in [25] that every regular cone is normal.

In the following, we always suppose that E is a Banach space, P is a cone in E with Int $P \neq \varnothing$ and $\leq$ is partial ordering with respect to P.

Definition 1 ([5]). Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d_1) $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if $x = y$,
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

Example 1 ([5]). Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2 ([5]). Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is N such that, for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is N such that, for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X. (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 1 ([5]). *Let (X, d) be a cone metric space, P be a normal cone and let $\{x_n\}$ be a sequence in X. Then*

(i) *$\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$ $(n \to \infty)$,*
(ii) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ $(n, m \to \infty)$.*

Let (X, d) be a cone metric space, $f : X \to X$ and $x_0 \in X$. Then the function f is continuous at x_0 if for any sequence $x_n \to x_0$ we have $fx_n \to fx_0$ [6].

2. Fixed point theorems for nondecreasing mappings

We begin by proving the following lemma. We can find the metric version of it in [33].

Lemma 2. *Let (X, d) be a cone metric space with the Banach space E, P be a cone in E, "$\leq$" be a partial ordering with respect to P and $\phi : X \to E$. Define the relation "$\preceq$" on X as follows:*

$$x \preceq y \Longleftrightarrow d(x, y) \leq \phi(x) - \phi(y).$$

Then "$\preceq$" is a (partial) order on X, named the partial order induced by ϕ.

Proof. For all $x \in X$, $d(x, x) = \theta = \phi(x) - \phi(x)$; that is, "$\preceq$" is reflexive. Again, for $x, y \in X$, let $x \preceq y$ and $y \preceq x$. Then,

$$d(x, y) \leq \phi(x) - \phi(y)$$

and

$$d(y, x) \leq \phi(y) - \phi(x).$$

This shows that $d(x, y) = \theta$; that is, $x = y$. Thus "$\preceq$" is antisymmetric. Now for $x, y, z \in X$, let $x \preceq y$ and $y \preceq z$. Then,

$$d(x, y) \leq \phi(x) - \phi(y) \tag{2.1}$$

and

$$d(y, z) \leq \phi(y) - \phi(z). \tag{2.2}$$

Then, using (2.1) and (2.2) we have

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &\leq \phi(x) - \phi(y) + \phi(y) - \phi(z) \\ &= \phi(x) - \phi(z). \end{aligned}$$

This shows that $x \preceq z$. □

Now we give some examples.

Example 2. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geq 0\}$, $X = \{a, b, c\}$ and $d : X \times X \to E$ such that $d(x, x) = (0, 0)$ for all $x \in X$, $d(a, b) = d(b, a) = (1, 2)$, $d(a, c) = d(c, a) = (1, 3)$ and $d(b, c) = d(c, b) = (2, 3)$. Then it is obvious