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A note on algebraic multigrid methods for the discrete weighted Laplacian

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ABSTRACT

In recent contributions, algebraic multigrid methods have been designed and studied from the viewpoint of spectral complementarity. In this note, we focus our efforts on specific applications and, more precisely, on large linear systems arising from the approximation of the weighted Laplacian with various boundary conditions. We adapt the multigrid idea to this specific setting and we present and critically discuss a wide set of numerical experiments showing the potentiality of the considered approach.

1. Introduction

In this note, we test a specific application of a previously proposed algebraic multigrid procedure [1]. In that work, we posed and partially answered the following question: having at our disposal an optimal multigrid procedure for $A_n x = b$, with$\{A_n\}$ being a given sequence of Hermitian positive definite matrices of increasing dimension, what are the minimal changes (if any) to the procedure for maintaining the optimality for $B_n y = c$, $\{B_n\}$ being a new sequence of matrices, with $B_n = A_n + R_n$?

Of course if there is no relation between $\{A_n\}$ and $\{B_n\}$ nothing can be said. However, under the mild assumption that there exists a value $\vartheta > 0$ independent of n such that $A_n \leq \vartheta B_n$ for $n \geq \bar{n}$ and $B_n \leq MI_n$ for $n \geq \bar{n}$ with M again independent of n, it has been shown that the smoothers can be easily adapted and the prolongation and restriction operators can be substantially kept unchanged. Here, the notation $X \leq Y$, with X and Y Hermitian matrices, means that $Y - X$ is non-negative definite.

The aim of this paper is to show the effectiveness of this approach in a specific setting. More precisely, we consider linear systems $A_n(a)u = b$ arising from finite difference (FD) approximations of

$$-\nabla(a(x)\nabla u(x)) = f(x), \quad x \in \Omega = (0,1)^d, \ d \geq 1,$$

where $a(x) \geq a_0 > 0$, $f(x)$ are given bounded functions and with Dirichlet boundary conditions (BCs). Some remarks about the case of periodic or reflective BCs are also considered (for a discussion on this topic see [2,3]).

We recall that when $a(x) \equiv 1$, the matrix $A_n(1)$ is structured, positive definite, and ill-conditioned, and an optimal algebraic multigrid method is already available (see [4–14]) according to different BCs.

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Hereafter, owing to the spectral equivalence between the matrix sequences $\{A_n(a)\}$ and $\{A_n(1)\}$, the key idea is that the multigrid procedure just devised for $\{A_n(1)\}$ can be successfully applied to $\{A_n(a)\}$ too. Here, $\{X_n\}$ and $\{Y_n\}$ are said to be spectrally equivalent if all X_n and Y_n are Hermitian and there exist constants $m, M > 0$, independent of n, such that $mX_n \le Y_n \le MX_n$ for each $n > 0$.

More generally, in [1], we treated the case of structured-plus-banded uniformly bounded Hermitian positive definite linear systems, where the banded part R_n which is added to the structured coefficient matrix A_n is not necessarily definite and not necessarily structured. Now, in our setting, $A_n = A_n(1)$ is the structured part (e.g., Toeplitz, circulant, etc., according to the BCs) and $R_n = A_n(a - 1)$ is the non-structured, not necessarily definite contribution.

However, while a theoretical analysis of the two-grid method (TGM) for structured-plus-banded uniformly bounded Hermitian positive definite linear systems has been given in [1], in terms of the algebraic multigrid theory due to Ruge and Stüben [15], the corresponding analysis for the multigrid method (MGM) is not complete and deserves further attention. Here, for the MGM algorithm, we mean the simplest (and less expensive) version of the large family of multigrid methods, i.e., the V-cycle procedure: for a brief description of the TGM and of the V-cycle algorithms we refer to Section 2, while an extensive treatment can be found in [16], and especially in [17].

Indeed, the numerics in this note suggest that the MGM is optimal in the sense that (see [18]) the cost of solving the linear system (inverse problem) is proportional, by a pure constant not depending on n, to the cost of the matrix–vector product (direct problem). In our case, more details can be given, and in fact

a. the observed number of iterations is bounded by a constant that is independent of the size of the algebraic problem;
b. the cost per iteration (in terms of arithmetic operations) is just linear with respect to the size of the algebraic problem.

Furthermore, given the spectral equivalence between $\{A_n(a)\}$, $a(x) \ge a_0 > 0$, and $\{A_n(1)\}$, a simpler numerical strategy could be used: use $A_n(1)$ as the preconditioner for $A_n(a)$ in a preconditioned conjugate gradient (PCG) method and solve the linear systems with coefficient matrix $A_n(1)$ by the MGM. Of course, this approach, which has been proposed, theoretically studied, and numerically validated in [19,12], is simpler to implement. However, several linear systems have to be solved by the MGM (one MGM application for every PCG step), while the design of an ad hoc MGM procedure implies the use on a single MGM application. There, when the coefficient a shows large jumps and is (close to) degenerate, the number of PCG iterations can become large, and consequently the flop count can be more favorable in applying a single MGM directly, instead of using it as solver for the preconditioner.

The paper is organized as follows. In Section 2 we report the standard TGM and MGM algorithms, together with the reference theoretical results on the TGM optimal rate of convergence, under some general and weak assumptions. In Section 3, the proposed approach is applied to the discrete weighted Laplacian and several numerical experiments are considered, by varying the diffusion coefficient $a(x)$ with respect to its analytical features. Finally, Section 4 deals with further considerations concerning future work and perspectives.

2. Two-grid and multigrid method

We carefully report the TGM and MGM algorithms and we describe the theoretical ground on which we base our proposal. We start with the simpler TGM algorithm and then we describe the MGM algorithm; its interpretation as stationary or multi-iterative method is also considered; see [20].

2.1. Algorithm definition

Let n_0 be a positive d-index, $d \ge 1$, and let $N(\cdot)$ be an increasing function with respect to n_0. In devising a TGM, and an MGM, for the linear system $A_{n_0}x_{n_0} = b_{n_0}$, where $A_{n_0} \in \mathbb{C}^{N(n_0)\times N(n_0)}$ and $x_{n_0}, b_{n_0} \in \mathbb{C}^{N(n_0)}$, the ingredients below must be considered.

Let $n_1 < n_0$ (componentwise) and let $p_{n_0}^{n_1} \in \mathbb{C}^{N(n_0)\times N(n_1)}$ be a given full-rank matrix. In order to simplify the notation, in the following we will refer to any multi-index n_s by means of its subscript s, so that, for example, $A_s := A_{n_s}$, $b_s := b_{n_s}$, $p_s^{s+1} := p_{n_s}^{n_{s+1}}$, etc.

With the same notations, a class of stationary iterative methods of the form $x_s^{(j+1)} = V_s x_s^{(j)} + \tilde{b}_s$ is also considered in such a way that $\mathcal{S}mooth(x_s^{(j)}, b_s, V_s, \nu_s)$ denotes the application of this rule ν_s times, with ν_s a positive integer number, at the dimension corresponding to the index s.

Thus, the solution of the linear system $A_0x_0 = b_0$ is obtained by applying repeatedly the TGM iteration, where the jth iteration

$$x_0^{(j+1)} = \mathcal{TGM}(x_0^{(j)}, b_0, A_0, V_{0,\mathrm{pre}}, \nu_{0,\mathrm{pre}}, V_{0,\mathrm{post}}, \nu_{0,\mathrm{post}})$$