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On solutions of set-valued optimization problems ☆

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ABSTRACT

We consider two criteria of a solution associated with a set-valued optimization problem, a vector criterion and a set criterion. We show how solutions of a vector type can help to find solutions of a set type and reciprocally. As an application, we obtain a sufficient condition for the existence of solutions of a set type via vector optimization.

1. Introduction

During the last decades, the set-valued optimization theory and its applications have been investigated by many authors; see [1–3] and the references therein.

The general expression of a set-valued optimization problem is

$$(\mathrm{P}) \quad \begin{cases} \text{Minimize } F(x) \\ \text{subject to } x \in X \end{cases}$$

where F is a set-valued map from a nonempty set X to a linear space Y ordered by a convex cone $K \subset Y$. For a problem (P) there exist two types of solutions: vector solutions, given by a vector criterion, and set solutions, given by a set criterion.

Set-valued optimization problems considering the vector criterion (or the standard notion) are called vector set-valued optimization problems and have been studied in various frameworks, for instance, see [4,2,5] and the references therein. This solution criterion cannot be the appropriate criterion when the decision maker's preference is based on comparing all image sets. It is just what the set criterion does. So, a less standard but perhaps more natural solution criterion was proposed: the set criterion (see [6]). Since then several authors have presented existence conditions for solutions of a set type; see [7–11].

Both solution concepts are entirely different and extend the concept of solution of a vector optimization problem. The main goal of this paper is to show some links between their solutions.

With this aim, firstly we will present some notations and definitions. In Section 3 we will establish conditions of alternative type which show how to obtain solutions of one type assuming that there are no solutions of another type and vice versa. Finally, a sufficient condition for the existence of solutions of a set type is given. We will show several examples to illustrate that the assumptions cannot be strong or difficult to be checked.

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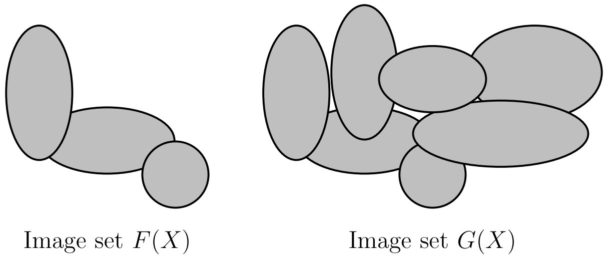

Fig. 1. Two examples of image sets.

2. Notations and definitions

Throughout this paper, X denotes a nonempty set of a topological space E, Y a separated topological linear space, $\wp_0(Y)$ the collection of nonempty subsets of Y and $K \subset Y$ a pointed ($K \cap -K = \{0\}$) solid closed convex cone. If $y,\ y' \in Y$ we denote by $y \leq y'$ if and only if $y' - y \in K$ and $y < y'$ if and only if $y \leq y'$ and $y \neq y'$.

Given a set $A \in \wp_0(Y)$, we denote by $Y \setminus A$ the complementary of A, by $\mathrm{int}(A)$ the topological interior of A, by $\partial(A)$ the boundary of A, by $\mathrm{Min}A = \{y \in A\colon (y-K)\cap A = \{y\}\}$ the set of minimal points of A and by $\mathrm{WMin}A = \{y \in A\colon (y-\mathrm{int}(K))\cap A = \emptyset\}$ the set of weakly minimal points of A.

Given a net $\{A_\alpha\}_{\alpha\in I}$ in $\wp_0(Y)$ where $(I, <)$ is a directed set we denote by $\liminf A_\alpha$ the set of all limit points of $\{A_\alpha\}_{\alpha\in I}$ and $\limsup A_\alpha$ the set of all cluster points of $\{A_\alpha\}_{\alpha\in I}$. It is clear that $\liminf A_\alpha \subset \limsup A_\alpha$.

A is K-minimal (or externally stable) if $A \subset \mathrm{Min}A + K$ (for more details see [4,12]).

We denote by $F\colon X \longrightarrow 2^Y$ a set-valued map with nonempty values, we write the image set of $A \subset X$ under F by $F(A) = \bigcup_{a\in A} F(a)$ and $\mathrm{Gr}(F) = \{(x, y) \in X \times Y\colon x \in X, y \in F(x)\}$. Whenever "N" denotes some property of sets in Y, it is said that F is "N"-valued if $F(x)$ has the property "N" for every $x \in X$. It is said that F is strongly injective if $F(x) \cap F(x') = \emptyset$ for all $x \neq x'$ and F is closed if $\mathrm{Gr}(F)$ is closed.

From now on, by (V-P) we denote the problem (P) using the following vector criterion of solution. We recall some classical definitions (see [4,2] and references therein).

Definition 2.1. Let $x_0 \in X$. It is said that x_0 is

(i) an efficient solution of (V-P), $x_0 \in \mathrm{Eff}[F, X]$, if there exists $y_0 \in F(x_0)$ such that $y_0 \in \mathrm{Min}\, F(X)$. The pair (x_0, y_0) is called minimizer of (V-P),

(ii) a weakly efficient solution of (V-P), $x_0 \in \mathrm{WEff}[F, X]$, if there exists $y_0 \in F(x_0)$ such that $y_0 \in \mathrm{WMin}F(X)$. The pair (x_0, y_0) is called weakly minimizer of (V-P).

We point out that the problem (V-P) always can be solved through a vector problem by minimizing the following projection map $\Pi_Y\colon \mathrm{Gr}(F) \longrightarrow Y$.

Now we show a geometric aspect of the vector criterion.

Example 2.2. Consider $Y = \mathbb{R}^2$ ordered by $K = \mathbb{R}^2_+$ and two set-valued maps F and G from X to Y such that their image sets, $F(X)$ and $G(X)$, are represented in Fig. 1. Then, in terms of minimal points, the vector criterion does not distinguish between both image sets.

To present the set criterion it is necessary to consider a relation between nonempty sets. In this paper, we focus on the following ones, if $A, B \in \wp_0(Y)$

$$A \leq^l B \quad \text{if and only if} \quad B \subset A + K$$

and

$$A \ll^l B \quad \text{if and only if} \quad B \subset A + \mathrm{int}(K).$$

The set relation $\leq^l$ was presented by the first time in the framework of linear spaces in [13] and the weak set relation $\ll^l$ was introduced in [14].

It is clear that the set relation $\sim^l$ defined as $A \sim^l B$ if and only if $A \leq^l B$ and $B \leq^l A$ is an equivalent relation on $\wp_0(Y)$. We denote by $[A]^l$ the equivalence class defined by A and by $\wp_0(Y)/\sim^l$ the quotient set formed by the equivalence classes of the relation $\sim^l$ on the elements of $\wp_0(Y)$.

By using the above set relations, we say that a net $\{A_\alpha\}_{\alpha\in I}$ in $\wp_0(Y)$ is l-decreasing if $\alpha > \beta$ implies that $A_\alpha \leq^l A_\beta$ and $A_\alpha \not\sim^l A_\beta$.

Let $\mathcal{S} \subset \wp_0(Y)$. We denote by l-$\mathrm{Min}\mathcal{S} = \{A \in \mathcal{S}\colon B \in \mathcal{S}$ and $B \leq^l A$ imply $A \leq^l B\}$ the family of l-minimal elements of $\mathcal{S}$ and by l-$\mathrm{WMin}\mathcal{S} = \{A \in \mathcal{S}\colon B \in \mathcal{S}$ and $B \ll^l A$ imply $A \ll^l B\}$ the family of weakly l-minimal elements of $\mathcal{S}$.