Computers and Mathematics with Applications 60 (2010) 1457–1464

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On Poncelet's maps

Anna Cima [a], Armengol Gasull [a,*], Víctor Mañosa [b]

[a] Dept. de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

[b] Dept. de Matemàtica Aplicada III (MA3), Control, Dynamics and Applications Group (CoDALab), Universitat Politècnica de Catalunya (UPC), Colom 1, 08222 Terrassa, Spain

ARTICLE INFO

Article history:
Received 4 August 2009
Received in revised form 17 June 2010
Accepted 17 June 2010

Keywords:
Poncelet's problem
Circle maps
Rotation number
Devil's staircase

ABSTRACT

Given two ellipses, one surrounding the other one, Poncelet introduced a map P from the exterior one to itself by using the tangent lines to the interior ellipse. This procedure can be extended to any two smooth, nested and convex ovals and we call these types of maps, Poncelet's maps. We recall what he proved around 1814 in the dynamical systems language: In the two ellipses' case and when the rotation number of P is rational there exists an $n \in \mathbb{N}$ such that $P^n = \text{Id}$, or in other words, Poncelet's map is conjugate to a rational rotation. In this paper we study general Poncelet's maps and give several examples of algebraic ovals where the corresponding Poncelet's map has a rational rotation number and is not conjugate to a rotation. Finally, we also provide a new proof of Poncelet's result based on dynamical and computational tools.

1. Introduction and main results

Let γ and Γ be two $\mathcal{C}^r$, $r \geq 1$, simple, closed and nested curves, each one of them being the boundary of a convex set. Furthermore we assume for instance that Γ surrounds γ.

Given any $p \in \Gamma$ there are exactly two points q_1, q_2 in γ such that the lines pq_1, pq_2 are tangent to γ. We define *Poncelet's map*, $P : \Gamma \to \Gamma$, associated to the pair γ, Γ as

$$P(p) = P_{\gamma,\Gamma}(p) = \overline{pq_1} \cap \Gamma,$$

where $p \in \Gamma$, $\overline{pq_1} \cap \Gamma$ is the first point in the set $\{\overline{pq_1} \cap \Gamma, \overline{pq_2} \cap \Gamma\}$ that we find when, starting from p, we follow Γ counterclockwise, see Fig. 1. Notice that $P^{-1}(p) = \overline{pq_2} \cap \Gamma$.

The implicit function theorem together with the geometrical interpretation of the construction of P imply that it is a $\mathcal{C}^r$ diffeomorphism from Γ into itself. So P can be seen as a $\mathcal{C}^r$ diffeomorphism of the circle and has associated a *rotation number*

$$\rho = \rho(P) = \rho(\gamma, \Gamma) \in (0, 1/2).$$

See for instance [1,2] for the definition of rotation number. Notice that usually a rotation number is in $(0, 1)$. Our choice of q_1 for Poncelet's map implies that indeed $\rho < 1/2$. A well known result of Denjoy states that if Φ is any diffeomorphism of the circle of class at least $\mathcal{C}^2$ and such that $\rho(\Phi) \notin \mathbb{Q}$ then Φ is conjugate to a rotation of angle $2\pi\rho(\Phi)$, see [3, p. 107] or [4, p. 45] for instance. So this is the situation for Poncelet's map P when $\rho(P) \notin \mathbb{Q}$ and $r \geq 2$.

With the above notation the celebrated *Poncelet's Theorem* asserts that if γ and Γ are ellipses, with arbitrary relative positions, and $\rho = \rho(\gamma, \Gamma) \in \mathbb{Q}$ then Poncelet's map is also conjugate to the rotation of angle $2\pi\rho$ in $\mathbb{S}^1$. In geometrical

* Corresponding author. Tel.: +34 93 5812909; fax: +34 93 5812790.
E-mail addresses: cima@mat.uab.cat (A. Cima), gasull@mat.uab.cat (A. Gasull), victor.manosa@upc.edu (V. Mañosa).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2010.06.027

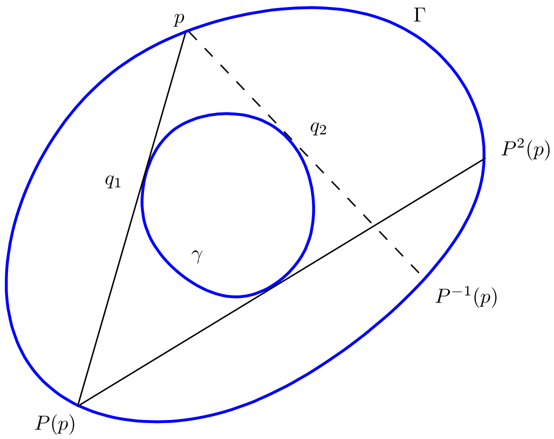

Fig. 1. Poncelet's map.

terms, if starting at some point $p \in \Gamma$ Poncelet's process of drawing tangent lines to γ closes after n steps then the same holds for any other starting point in Γ. There are several proofs of this nice result in [5, Sec. 4.3] and a different one, based on a beautiful approach of Bertrand and Jacobi through differential equations and elliptic integrals in [6, pp. 191–194]. In Section 4 we give another proof based on dynamical and computational tools, by using the results of [7]. The problem of determining explicit conditions over the coefficients of the two ellipses to ensure that Poncelet's map is conjugate to a rational rotation was solved by Cayley. An excellent exposition of this result is given in [8].

A monograph devoted to Poncelet's Theorem and related results has recently appeared, see [9].

It is clear that in Poncelet's Theorem it is not restrictive to assume that $\gamma = \{x^2 + y^2 = 1\}$. The first question that we face in this paper is the following: Is Poncelet's result also true when we consider $\gamma = \{x^2 + y^2 = 1\}$ and Γ given by an oval of an algebraic curve of higher degree? We prove:

Theorem 1. *Fix $\gamma = \{x^2 + y^2 = 1\}$. Then for any $m \in \mathbb{N}$, $m > 2$, there is an algebraic curve of degree m, containing a convex oval Γ, such that Poncelet's map associated to γ and Γ has rational rotation number and it is not conjugate to a rotation.*

This result will be a consequence of a more general result proved in Section 2, see Proposition 2. Moreover the full dynamics of the introduced Poncelet's maps $P : \Gamma \to \Gamma$ will be described in that section.

From Theorem 1 it is clear that, in general, Poncelet's maps with rational rotation numbers are not conjugate to rotations. It is natural to wonder about this question when both ovals are level sets of the same polynomial map $V : \mathbb{R}^2 \to \mathbb{R}$. As one of the simplest cases we consider the homogeneous map $V(x, y) = x^4 + y^4$, giving

$$\gamma = \Gamma_1 = \{x^4 + y^4 = 1\} \quad \text{and} \quad \Gamma = \Gamma_k = \{x^4 + y^4 = k\},$$

for $k \in \mathbb{R}$, $k > 1$. As we will see in Section 3 even in this situation the conjugacy with a rotation is not true.

As usual a planar map Φ will be called *integrable* if there exists a non-constant real valued function V such that the orbits generated by Φ are contained in its levels sets. The function V is called a *first integral* of Φ and it holds that $V(\Phi) = V$.

Poncelet's maps also provide a natural way of defining an integrable map from an open set of $\mathbb{R}^2$ into itself as follows: We foliate the open unbounded connected component $\mathcal{U}$ of $\mathbb{R}^2 \setminus \Gamma_1$ as

$$\mathcal{U} := \bigcup_{k>1} \{x^4 + y^4 = k\}.$$

Then Poncelet's construction gives a new diffeomorphism, also of class $\mathcal{C}^r$, that is defined from $\mathcal{U} \subset \mathbb{R}^2$ into itself, which simply consists in applying to each point p the corresponding Poncelet's map, associated to the level set of V passing through p. For the sake of simplicity we also call it P. Notice that this map is trivially integrable by means of $V(x, y) = x^4 + y^4$, that is $V(P(x, y)) = V(x, y)$ for all $(x, y) \in \mathcal{U}$.

As we will see in Section 3.4, this extended Poncelet's map P will be useful to give properties of a functional equation that helps to study integrable planar maps.

Finally, in the above context it is natural to introduce the rotation function $\rho(k) := \rho(\Gamma_1, \Gamma_k)$, as the rotation number of P associated to γ and Γ_k, and to study some of its properties.

In the case of two concentric circles

$$\gamma = \widetilde{\Gamma}_1 = \{x^2 + y^2 = 1\} \quad \text{and} \quad \Gamma = \widetilde{\Gamma}_k = \{x^2 + y^2 = k\},$$

it is easy to prove that the rotation function $\widetilde{\rho}(k) = \rho(\widetilde{\Gamma}_1, \widetilde{\Gamma}_k)$ is the smooth monotonous function $\widetilde{\rho}(k) = \arctan(k-1)/\pi$. On the other hand, in Section 3 we will show that the function $\rho(k) := \rho(\Gamma_1, \Gamma_k)$, is much more complicated. Indeed all what we prove seems to indicate that it has the usual shape of the rotation function of generic one-parameter families of diffeomorphisms: the devil's staircase, see for instance [10,11].