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Convergence analysis of an iterative scheme for solving initial value problem for multidimensional partial differential equations

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ABSTRACT

Existence and uniqueness of solutions of initial value problem for certain classes of multidimensional linear and nonlinear partial differential equations are proved using Banach fixed-point theorem. An iterative scheme is derived and rigorous convergence analysis of this scheme and an error estimate are included as well. Several numerical examples for high dimensional initial value problem for heat and wave type partial differential equations are presented to demonstrate reliability and performance of proposed iterative scheme.

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1. Introduction

Many physical phenomena can be described by mathematical models that involve partial differential equations. Therefore, in recent years, researchers look for new methods which are cost efficient and simple in implementation to solve partial differential equations. Investigation of exact and approximate solution helps us to understand meaning and relevance of these mathematical models. Several techniques including scattering method [1], sine–cosine method [2], homotopy analysis method [3], homotopy perturbation method [4–8], differential transform method [9], variational iteration method [10], or decomposition methods [11–15] have been recently used for solving these problems, but mostly for two dimensional partial differential equations. However, most of the above mentioned methods need some analytical manipulation before computer software can be applied. Some of the algorithms require software using symbolical integration.

Inspired and motivated by ongoing research in this area, we prove local existence and uniqueness of solution of considered problem using Banach fixed-point theorem and then using this result we derive an iterative scheme for solving initial problems for heat- and wave-type equations. Several examples are given to verify reliability and efficiency of proposed scheme.

The paper is organized as follows. In Section 2, all necessary notations, definitions and conditions are formulated. This proper preparation enables us to prove the main results as well as corollaries in Section 3. Section 4 consists of several explanatory examples. Conclusions are summarized in Section 5.

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2. Preliminaries

Let Ω be a compact subset of $\mathbb{R}^k$. Denote $J = [-\delta, \delta] \times \Omega$, where $\delta > 0$ will be specified later, then J is a compact subset of $\mathbb{R}^{k+1}$. Let $u(t, x) = u(t, x_1, \dots, x_k)$ be a real function of $k+1$ variables defined on J. We introduce the following operators: $\nabla = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k})$ and $D = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k})$. We deal with partial differential equations

$$\frac{\partial^n}{\partial t^n} u(t, x) = F(t, x, u, \nabla u, \dots, \nabla^m u) \quad \text{for } m < n \tag{1}$$

and

$$\frac{\partial^n}{\partial t^n} u(t, x) = F(t, x, u, \nabla u, \dots, \nabla^{n-1} u, D\nabla^{n-1} u, D^2\nabla^{n-1} u, \dots, D^{m-(n-1)}\nabla^{n-1} u) \quad \text{for } m \geq n. \tag{2}$$

In both cases, left-hand side of the equation contains only the highest derivative with respect to t. We do not consider equations where the order of partial derivatives with respect to t is n or higher on the right hand side, including mixed derivatives.

When convenient, we will use multiindex notation as well:

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_k^{\alpha_k}},$$

where $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_k$.

Denote $N = \max\{m, n\}$. We consider Eqs. (1) or (2) with the set of initial conditions

$$\begin{aligned} &u(0, x) = c_1(x), \\ &\frac{\partial}{\partial t} u(0, x) = c_2(x), \\ &\vdots \\ &\frac{\partial^{n-1}}{\partial t^{n-1}} u(0, x) = c_n(x), \end{aligned} \tag{3}$$

where initial functions $c_i(x),\ i = 1, \dots, n$ are taken from space $C^N(\Omega, \mathbb{R})$. It means that we are looking for classical solutions.

For the purpose of clarity, we emphasize that our formulation covers for instance heat, wave, Burger, Boussinesq or Korteweg–de Vries (KdV) equations.

Obviously, $F: J \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{k+1} \times \mathbb{R}^{(k+1)^2} \times \dots \times \mathbb{R}^{(k+1)^m} \to \mathbb{R}$ if $m < n$, and $F: J \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{k+1} \times \mathbb{R}^{(k+1)^2} \times \dots \times \mathbb{R}^{(k+1)^{n-1}} \times \mathbb{R}^{k(k+1)^{n-1}} \times \mathbb{R}^{k^2(k+1)^{n-1}} \times \dots \times \mathbb{R}^{k^{m-n+1}(k+1)^{n-1}} \to \mathbb{R}$ if $m \geq n$. Denote

$$K_1 = \frac{(k+1)^{m+1} - 1}{k} \quad \text{for } m < n \tag{4}$$

and

$$K_2 = \frac{(k+1)^{n-1} - 1}{k} + (k+1)^{n-1} \frac{k^{m-n+2} - 1}{k-1} \quad \text{for } m \geq n. \tag{5}$$

Then, if we consider u as dependent variable, we see that F is a function of $k + 1 + K_1$ variables in case $m < n$ or $k + 1 + K_2$ variables in case $m \geq n$.

Denote

$$u_0(t, x) = \sum_{i=1}^{n} c_i(x) \frac{t^{i-1}}{(i-1)!} = \sum_{i=1}^{n} \left(\frac{\partial^{i-1}}{\partial t^{i-1}} u(0, x) \right) \frac{t^{i-1}}{(i-1)!}. \tag{6}$$

Then $u_0 \in C^N(J, \mathbb{R})$.

We suppose that F is Lipschitz continuous in last K_1 ($m < n$) or K_2 ($m \geq n$) variables, i.e. F satisfies condition

$$|F(t, x, y_1, \dots, y_{K_l}) - F(t, x, z_1, \dots, z_{K_l})| \leq L \left(\sum_{i=1}^{K_l} |y_i - z_i| \right), \quad l = 1 \text{ or } 2, \tag{7}$$

on a compact set which is defined as follows: There is $R \in \mathbb{R}$, $R > 0$ such that (7) holds on

$$J \times \prod_{\alpha_0 + |\alpha| \leq m} [c_{\alpha_0, \alpha}, d_{\alpha_0, \alpha}], \tag{8}$$

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