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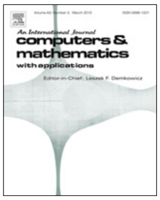

Stationary problem of a predator–prey system with nonlinear diffusion effects

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ABSTRACT

In this paper, we are concerned with the positive solution set of the following quasilinear elliptic system

$$\begin{cases} -\Delta\left[(1+\alpha v)\,u\right] = u\left(a-u-\dfrac{cv}{1+mu}\right), & x\in\Omega, \\ -\Delta\left[\left(1+\dfrac{\gamma}{1+\beta u}\right)v\right] = v\left(b-v+\dfrac{du}{1+mu}\right), & x\in\Omega, \\ u=v=0, & x\in\partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, α, β, γ are nonnegative constants, a, c, d, m are positive constants, b is a real constant. This elliptic system is the stationary problem of a predator–prey model, in which u and v denote the population densities of the prey and predator, respectively. Regarding b as the bifurcation parameter, the global bifurcation structure of the positive solution set is shown. Then the nonlinear effect of either large α or β on the bifurcation point is deduced. Moreover, as α is large, we show that the positive solution set is only of one type, whereas, as β is large, the positive solution set is of two types. Additionally, in certain circumstances, we also show that which one of the two types can characterize the positive solution set.

1. Introduction

In this paper, we study the stationary problem of the following predator–prey system with nonlinear diffusion terms:

$$\begin{cases} u_t = \Delta\left[(1+\alpha v)\,u\right] + u\left(a-u-\dfrac{cv}{1+mu}\right), & (x,t)\in\Omega\times(0,T), \\ v_t = \Delta\left[\left(1+\dfrac{\gamma}{1+\beta u}\right)v\right] + v\left(b-v+\dfrac{du}{1+mu}\right), & (x,t)\in\Omega\times(0,T), \\ u=v=0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x)\ge 0, \qquad v(x,0)=v_0(x)\ge 0, & x\in\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. The coefficients a, c, d and m are all positive constants, while b is a real constant and may be negative. The coefficients α, β and γ are nonnegative constants. From the ecological viewpoint, the functions u and v denote the population densities of the prey and predator, respectively; a and b are the growth rates of the prey and predator, respectively. The interaction between the prey and the predator is the Holling-II functional response $u/(1 + mu)$, which is most commonly used in the literature.

The diffusion terms here reflect the fact that the diffusion of the species is affected by its interacting species. In the model (1.1), we can see that the diffusion rate of the prey species increases when the predator species appears, while the diffusion rate of the predator species decreases in the presence of the prey species but never becomes negative. Thus, the prey species moves more rapidly to try to avoid the predator species when the predator species is around, but the predator species moves more slowly to try to stay near the prey species in the presence of the prey species [1]. Thus, the cross-diffusion term is in agreement with the general observed pattern of the interaction between the prey and predator. So it is realistic. Then it is quite an interesting problem to study its stationary problem and show the effects of the cross-diffusion on the stationary problem.

Since the pioneering work by Shigesada, Kawasaki and Teramoto [2], many researchers have been devoted to the study of the population models with cross-diffusion from various mathematical viewpoints, one can refer to [3–14] and references therein. However, there are few works on the nonlinear diffusion of this fractional type due to its complexity. As far as we know, under Neumann boundary condition, Wang and Li [15] studied a Holling–Tanner predator–prey model with this kind of cross-diffusion, and showed the existence and nonexistence of nonconstant positive stationary solutions. Under Dirichlet boundary condition, Kuto et al. have investigated the positive stationary solution of the Lotka–Volterra predator–prey model with this kind of cross-diffusion in a series of his papers, see [16–19]; Chen and Peng [20] also did some work for the model studied by Kuto; considering the modified Leslie–Gower and Holling-II functional responses, Wu, Guo and Ma [21] studied a predator–prey model with this kind of cross-diffusion, and showed some results about its positive stationary solution.

It is clear that the stationary problem of system (1.1) is the following quasilinear elliptic system:

$$\begin{cases} -\Delta\left[(1+\alpha v)u\right] = u\left(a - u - \dfrac{cv}{1+mu}\right), & x \in \Omega, \\ -\Delta\left[\left(1 + \dfrac{\gamma}{1+\beta u}\right)v\right] = v\left(b - v + \dfrac{du}{1+mu}\right), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Due to the ecological meaning, only positive solutions of (1.2) are of interest. It is said that (u, v) is a positive solution if (u, v) satisfies (1.2) with $u, v > 0,\ x \in \Omega$. Thus, a positive solution (u, v) corresponds to a coexistence state of the prey species and predator species. When $\alpha = \beta = \gamma = 0$, system (1.2) is the classical predator–prey model with Holling-II functional response and has been studied extensively, one can refer to [22–25] and references therein. In [24], Du and Lou gave a clear description of the existence, stability and number of the positive solutions for large m. In [25], Du and Lou further showed the S-shaped global bifurcation curve and Hopf bifurcation of the positive solutions. For a detailed study of large m subject to homogeneous Neumann boundary condition, see their work [26].

In the present paper, we try to obtain some understanding of the structure of the positive solution set of (1.2). First, combining the bifurcation theory and some estimates, we take b as the bifurcation parameter and give the global bifurcation structure of the positive solution set of (1.2) bifurcating from its semitrivial solution sets Γ_u and Γ_v. Precisely, the positive solution set of (1.2) contains a bounded continuum, which connects the semitrivial solution set Γ_u at $b = b_*$ with the other semitrivial solution set Γ_v at $b = b^*$. Then, we investigate the effects of large cross-diffusion on the positive solution set. Our result shows that the bifurcation point $b = b_*$ becomes smaller as the cross-diffusion β is sufficiently large, and $b = b^*$ also becomes smaller as the cross-diffusion α is sufficiently large. Moreover, we characterize the type of the positive solution set of (1.2) as either β or α is large. To do so, we introduce the limiting system as the cross-diffusion β or α tends to infinity and give the global bifurcation structure of the positive solution set of the limiting system. In fact, two limiting systems are deduced as $\beta \to \infty$, while only one limiting system exists as $\alpha \to \infty$. Thus, in the former case, the positive solution set is of two types, that is, either (u, v) or $(\beta u, v)$ tends to a certain pair of positive functions. In the latter case, the positive solution set is of one type, that is, $(u, \alpha v)$ tends to a certain pair of positive functions. Thus, as $\beta \to \infty$, either $\|u\|_\infty = O(1)$ or $\|u\|_\infty = O(1/\beta)$; while as $\alpha \to \infty$, $\|v\|_\infty = O(1/\alpha)$. Furthermore, we can determine that which one of the two limiting systems characterizes the positive solution set of (1.2) as β is sufficiently large. As shown in Remark 5.6, if the predator has a small birth rate, then $\|u\|_\infty = O(1)$ and (u, v) is approximated by a certain pair of positive functions; whereas, if the predator has a comparatively large growth rate b, then $\|u\|_\infty = O(1/\beta)$ and $(\beta u, v)$ is approximated by a certain pair of positive functions.

From the global bifurcation of the positive solution set, we can deduce a coexistence region of (1.2), see Fig. 1. By the nonlinear effects of large cross-diffusion on the bifurcation point, we can see that the coexistence region obtained in this paper may become larger as β is sufficiently large and smaller as α is sufficiently large, see Figs. 2 and 3. But whether the coexistence region is the exact coexistence region or not needs a further and careful study.

It should be pointed out that a priori estimate of the positive solutions of (1.2) plays an essential role in this paper. First, a priori estimate of the positive solutions independent of β can be obtained by the maximum principle. As a result, there is no restriction on the spatial dimension N in the study of the limiting behavior of the positive solutions as $\beta \to \infty$. However,