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Fixed point theorems for generalized contractive multi-valued maps

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ABSTRACT

In [N. Mizoguchi, W. Takahashi, Fixed point theorems for multi-valued mappings on complete metric spaces, J. Math. Anal. Appl. 141 (1989) 177–188] the authors gave a positive answer to the conjecture of S. Reich concerning the existence of fixed points of multi-valued mappings that satisfy certain contractive conditions. In this paper, we establish some results for multi-valued mappings that satisfy a generalized contractive condition in a way that it contains Mizoguchi's result as one of its special cases. In addition, our results not only improve the results of Kiran and Kamran [Q. Kiran, T. Kamran, Nadler's type principle with high order of convergence, Nonlinear Anal. TMA 69 (2008) 4106–4120] and some results of Agarwal et al. [R.P. Agarwal, Jewgeni Dshalalow, Donal O'Regan, Fixed point and homotopy results for generalized contractive maps of Reich type, Appl. Anal. 82 (4) (2003) 329–350] but also provide the high order of convergence of the iterative scheme and error bounds. As an application of our results, we obtain an existence result for a class of integral inclusions.

1. Introduction and preliminaries

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $N(X)$ the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty bounded closed subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. Let H be the generalized Hausdorff metric on $CB(X)$ generated by the metric d, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}$$

for every $A, B \in CB(X)$. A point $p \in X$ is said to be a fixed point of $T : X \to CL(X)$ if $p \in Tp$. If, for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$ then $O(T, x_0) = \{x_0, x_1, x_2, \ldots\}$ is said to be orbit of $T : X \to CL(X)$. A mapping $f : X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous if $\{x_n\}$ is a sequence in $O(T, x_0)$ and $x_n \to \xi$ implies $f(\xi) \le \lim_n \inf f(x_n)$. Throughout this paper J denotes an interval on $\mathbb{R}_+$ containing 0, that is an interval of the form $[0, A]$, $[0, A)$ or $[0, \infty)$ and $S_n(t)$ denotes the polynomial $S_n(t) = 1 + t + \cdots + t^{n-1}$. We use the abbreviation φ^n for the nth iterate of a function $\varphi : J \to J$.

Definition 1.1 (*[1]*)**.** Let $r \ge 1$. A function $\varphi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:

(i) $\varphi(\lambda t) \le \lambda^r \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$;
(ii) $\varphi(t) < t$ for all $t \in J - \{0\}$.

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It is easy to see that the first condition of Definition 1.1 is equivalent to the following: $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J - \{0\}$. We are stating the following results for convenience.

Lemma 1.2 ([2]). *Let $A, B \in CB(X)$ and let $a \in A$. If $\epsilon > 0$, then there exists $b \in B$ such that $d(a, b) \le H(A, B) + \epsilon$.*

Lemma 1.3 ([1]). *Let φ be a gauge function of order $r \ge 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying*

$$\varphi(t) = t\phi(t) \quad \text{for all } t \in J, \tag{1}$$

then it has the following two properties:
(i) *$0 \le \phi(t) < 1$ for all $t \in J$;*
(ii) *$\phi(\lambda t) \le \lambda^{r-1}\phi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$.*

Lemma 1.4 ([1]). *Let φ be a gauge function of order $r \ge 1$ on J. Then for every $n \ge 0$ we have*
(i) *$\varphi^n(t) \le t\phi(t)^{S_n(r)}$ for all $t \in J$,*
(ii) *$\phi(\varphi^n(t)) \le \phi(t)^{r^n}$ for all $t \in J$,*
where ϕ is a nonnegative and nondecreasing function on J satisfying (1).

Definition 1.5 ([1]). A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini–Grandolfi gauge function [3] on J if

$$\sigma(t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \quad \text{for all } t \in J. \tag{2}$$

Note that Ptak [4] called a function $\varphi : J \to J$ satisfying (2) a rate of convergence on J and noticed that φ satisfies the following functional equation

$$\sigma(t) = \sigma(\varphi(t)) + t. \tag{3}$$

The following statement is an immediate consequence of the first part of Lemma 1.4 and the obvious inequality $S_n(r) \ge n$ for all $r \ge 1$.

Lemma 1.6 ([1]). *Every gauge function of order $r \ge 1$ on J is a Bianchini–Grandolfi gauge function on J.*

Definition 1.7 ([5]). Suppose (x_n) is a sequence that converges to ξ. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{d(x_{n+1}, \xi)}{(d(x_n, \xi))^{\alpha}} = \lambda$$

then (x_n) is said to converge to ξ of order α, with asymptotic error constant λ.

Remark 1.8. In general, a sequence with high order of convergence converges more rapidly than a sequence with a lower order. If $\alpha = 1$, the method is called linear. If $\alpha = 2$, the method is called quadratic.

In [6], Reich proved that a mapping $T : X \to K(X)$ has a fixed point in X if it satisfies

$$H(Tx, Ty) \le k(d(x, y))d(x, y) \tag{4}$$

for all $x, y \in X$ with $x \ne y$, where $k : (0, \infty) \to [0, 1)$ satisfies $\limsup_{s\to t^+} k(s) < 1$ for every $t \in (0, \infty)$. This result generalizes the fixed point theorem for single-valued mappings that was proved by Boyd and Wong [7]. Reich questioned in [8,9] that whether or not the range of T, $K(X)$ can be replaced by $CB(X)$. Mizoguchi and Takahashi [10], Daffer and Kaneko [11] and Tong-Huei Chang [12] gave a positive answer to the conjecture of Reich. Recently, Pathak and Shahzad [13] generalized Nadler's contraction principle in contrast to Reich's and Mizoguchi–Takahashi's theorems. More recently, Thagfi and Shahzad [14] obtained some fixed point theorems for an operator which is closely related to the Reich type contraction. The authors in [15] extended some results of Proinov [1] to the case of multi-valued maps from a complete metric space X into the space of all nonempty proximinal closed subsets of X. The purpose of this paper is to obtain some fixed point theorems for multi-valued maps which not only provide the iterative scheme with a high convergence rate but also the error bounds. Our results generalize [10, Theorem 5], [11, Theorem 2.1], [15, Theorems 2.11 & 2.15] and [16, Theorems 2.1 & 2.2].

2. Main results

Theorem 2.1. *Let (X, d) be a complete metric space, D be a closed subset of X, φ is a Bianchini–Grandolfi gauge function on an interval J and T be a mapping from D into $CB(X)$ such that $Tx \cap D \ne \emptyset$ and*

$$H(Tx \cap D, Ty \cap D) \le \varphi(d(x, y)) \tag{5}$$

for all $x \in D$, $y \in Tx \cap D$ with $d(x, y) \in J$. Moreover, the strict inequality holds when $d(x, y) \ne 0$. Suppose $x_0 \in D$ is such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$. Then:
(i) *there exists an orbit $\{x_n\}$ of T in D and $\xi \in D$ such that $\lim_n x_n = \xi$;*
(ii) *ξ is a fixed point of T if and only if the function $f(x) := d(x, Tx \cap D)$ is T-orbitally lower semi-continuous at ξ.*