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Positivity and boundedness preserving schemes for space–time fractional predator–prey reaction–diffusion model

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ABSTRACT

The semi-implicit schemes for the nonlinear predator–prey reaction–diffusion model with the space–time fractional derivatives are discussed, where the space fractional derivative is discretized by the fractional centered difference and WSGD scheme. The stability and convergence of the semi-implicit schemes are analyzed in the L_∞ norm. We theoretically prove that the numerical schemes are stable and convergent without the restriction on the ratio of space and time stepsizes and numerically further confirm that the schemes have first order convergence in time and second order convergence in space. Then we discuss the positivity and boundedness properties of the analytical solutions of the discussed model, and show that the numerical solutions preserve the positivity and boundedness. The presented numerical examples confirm the theoretical results and convergence orders.

1. Introduction

The predator–prey model, also known as the Lotka–Volterra equation, is a pair of first-order nonlinear ordinary differential equations frequently used to describe the dynamics of biological system in which two species interact, one as the predator and the other as prey [1]; and the unknown variables usually denote certain measure of total population. To interpret the unknown variables as spacial densities so as to allow the population size to vary throughout the considered region, Conway and Smoller introduce diffusion terms to the predator–prey model which allows diffusing as well as interacting each other [2]; and then the predator–prey reaction–diffusion model is obtained. With the development of the predator–prey model, it seems natural to introduce diffusion terms to the corresponding model, e.g., the Michaelis–Menten–Holling predator–prey reaction–diffusion model [3,4]; and more general other models [5–8].

Anomalous diffusion, including subdiffusion and superdiffusion, is also a diffusion process, but its mean squared displacement (MSD) is nonlinear with respect to time t, in contrast to the classical diffusion process, in which the MSD is linear [9,10]; nowadays, it is widely recognized that the anomalous diffusion is ubiquitous, e.g., diffusion through porous media, protein diffusion within cells, and also being found in many other biological systems. So it seems reasonable/natural to introduce the anomalous diffusion to the predator–prey model. The subdiffusion is introduced to the Michaelis–Menten–Holling predator–prey model in [11] to get a new model with time fractional derivative; it is proved that the solution of the model is positive and bounded; and the numerical schemes preserving the positivity and boundedness are detailedly discussed. Here we introduce both the subdiffusion and superdiffusion to the Michaelis–Menten–Holling predator–prey model. Fractional

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calculus plays important role in characterizing the time evolution of the probability density function (PDF) of anomalous diffusion; the Fokker–Planck equation with time fractional derivative describes the time evolution of the PDF of the subdiffusion particles; the Fokker–Planck equation with space fractional derivative characterizes the time evolution of the PDF of the superdiffusion particles; and the Fokker–Planck equation with both the time and space fractional derivatives describes the time evolution of the PDF of the subdiffusion/superdiffusion particles, which is a result of the competition of subdiffusion and superdiffusion [9]. So the system can be written as

$$\begin{aligned}
\frac{\partial^\alpha N}{\partial t^\alpha} &= D_1\frac{\partial^\beta N}{\partial |x|^\beta} + N\left(1 - N - \frac{\varrho P}{P+N}\right), \quad x\in(l,r),\ t>0,\\
\frac{\partial^\alpha P}{\partial t^\alpha} &= D_2\frac{\partial^\beta P}{\partial |x|^\beta} + \sigma P\left(-\frac{\gamma+\kappa\delta P}{1+\kappa P} + \frac{N}{P+N}\right), \quad x\in(l,r),\ t>0,
\end{aligned} \tag{1.1}$$

with the Caputo derivative in time and Riesz space fractional derivative, where ϱ, σ and κ are positive real numbers and N and P denote the population densities of prey and predator respectively. Based on the practical applications, we are interested in the solutions of (1.1) with the nonnegative initial conditions

$$N(x,0) = g_1(x) \ge 0, \qquad P(x,0) = g_2(x)\ge 0, \quad x\in(l,r), \tag{1.2}$$

and the homogeneous Neumann boundary conditions

$$\left.\frac{\partial N(x,t)}{\partial x}\right|_{x=l} = \left.\frac{\partial N(x,t)}{\partial x}\right|_{x=r} = \left.\frac{\partial P(x,t)}{\partial x}\right|_{x=l} = \left.\frac{\partial P(x,t)}{\partial x}\right|_{x=r} = 0, \quad 0<t\le T. \tag{1.3}$$

The positive constants γ and δ in the coupled equations denote the minimal mortality and the limiting mortality of the predator, respectively. Throughout the paper, we assume that γ satisfies the natural condition $0<\gamma\le\delta$ and consider the case of the diffusion constants $D_i>0,\ i=1,2$.

The numerical methods for solving fractional partial differential equations are developing fast; most of them focus on fractional diffusion equations, including the space fractional diffusion equation [12,13] and the time fractional diffusion equation [14–19]. Because of the stability issue, the space fractional derivative is usually approximated by the shifted Grünwald–Letnikov definition with the finite stepsize; and the truncation error is first-order. Recently, the second-order discretizations for space fractional derivative appear: based on the so-called "fractional centered difference", Ortigueira gets the second-order approximation for the Riesz fractional derivative [20], and its applications can be seen in [21,22]; Tian et al. obtain the so-called WSGD second order approximation for both the left and right Riemann–Liouville derivatives [23], and its compact version has third-order accuracy [24].

This paper first proves that the analytical solutions of the space–time fractional predator–prey reaction–diffusion model (1.1)–(1.3) are positive and bounded; and then designs the numerical schemes to solve it. The space fractional derivative is discretized by the fractional centered difference [20] and the WSGD operators [23], respectively. We prove that both of the two obtained schemes have second-order accuracy in space and preserve the positivity and boundedness of the analytical solutions. In particular, the stability of the numerical scheme without the restriction on the ratio of the space and time stepsizes is also strictly proved. And a numerical example is provided to confirm the theoretical results.

The outline of this paper is as follows. In Section 2, we present two finite difference schemes for the space–time fractional predator–prey reaction–diffusion model. The detailed discussions on the stability and convergence with first-order in time and second-order in space are given in Section 3. In Section 4, we prove that the analytical solutions of the discussed model are positive and bounded; and the two schemes preserve the positivity and boundedness. To confirm the convergence orders and the positivity and boundedness preserving, the numerical experiments are performed in Section 5. And we conclude the paper with some discussions in the last section.

2. Numerical schemes for the space–time fractional predator–prey reaction–diffusion model

In this section, we detailedly discuss the numerical schemes of the model (1.1), where the time fractional operator $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes the Caputo fractional derivative

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{1}{(t-s)^\alpha}ds,$$

and $\frac{\partial^\beta u}{\partial|x|^\beta}$ is the Riesz fractional derivative

$$\frac{\partial^\beta u(x,t)}{\partial|x|^\beta} = -\frac{1}{2\cos(\beta\pi/2)\Gamma(2-\beta)}\frac{d^2}{dx^2}\int_l^r |x-\xi|^{1-\beta}u(\xi,t)d\xi, \tag{2.1}$$

with $\alpha\in(0,1)$ and $\beta\in(1,2)$.

For ease of presentation, we uniformly divide the spacial domain $[l,r]$ into M_h subintervals with stepsize $h=(r-l)/M_h$ and the time domain $[0,T]$ into M_τ subintervals with steplength $\tau = T/M_\tau$. Let $x_i = l+ih\ (i=0,1,\ldots,M_h)$, $t_k = k\tau$