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Error estimates of the weakly over-penalized symmetric interior penalty method for two variational inequalities

Yuping Zeng [a,*], Jinru Chen [b], Feng Wang [b]

[a] School of Mathematics, Jiaying University, Meizhou 514015, China
[b] Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China

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ABSTRACT

In this paper, we apply the weakly over-penalized symmetric interior penalty method to solve some variational inequalities which include the Signorini problem and the obstacle problem. Optimal a priori error estimates in energy norm are derived. Some numerical tests are provided to confirm our theoretical analysis.

1. Introduction

As an important class of nonlinear problems, variational inequalities often arise from mechanics, physics and engineering science. For this reason, the mathematical theory and numerical analysis of variational inequalities have made impressive progress in the past four decades [1–3]. A prototype model involving a variational inequality is the Signorini problem, which models the unilateral contact. Another prototype model is the elliptic obstacle problem, which describes the membrane deformation phenomenon. The continuous finite element methods for solving both models have been studied extensively [4–7]. Details on a priori error estimates of continuous finite element methods for the Signorini problem can be found in [4,6–8]. We also refer readers to [6,9] for the a priori error estimates of continuous finite element methods for the obstacle problem. For the nonconforming finite element methods for solving variational inequalities, please see [10,11].

In this article, we are concerned with the development of a weakly over-penalized symmetric interior penalty (WOPSIP) method for both prototype models above. The WOPSIP method belongs to a class of discontinuous Galerkin (DG) methods, which was initially proposed in [12] by Brenner et al. to solve second order elliptic equations. In recent years, DG methods have received much interest due to their suitability for *hp*-adaptive techniques. Some further advantages of DG schemes are that they can easily handle inhomogeneous boundary conditions, curved boundaries, and highly nonuniform and unstructured meshes. In contrast to standard continuous finite elements, applying DG methods to solve variational inequalities happened in recent years. The first article concerning this issue is addressed in [13], where the symmetric and nonsymmetric interior penalty DG methods are proposed and analyzed. Later, a unified framework of some well known DG methods presented in [14] has been extended to solve variational inequalities [15,16]. For a detailed numerical analysis of local discontinuous Galerkin method for the Signorini problem, we refer to [17]. Our work will further study the WOPSIP methods

* Corresponding author.
E-mail address: zeng_yuping@163.com (Y. Zeng).

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for both Signorini problems and obstacle problems. From [12] we know that the WOPSIP method has many advantages, e.g., compared with many well-known DG methods, it has less computational complexity and it is easy to implement. In addition, it was shown that [18] the WOPSIP method has high intrinsic parallelism. For these reasons, the WOPSIP method has been further developed to solve non-self-adjoint and indefinite problems [19], biharmonic problems [20], Stokes equations [21] and Reissner–Mindlin plate equations [22]. The main objective of this paper is to give a detailed analysis of the WOPSIP DG method for Signorini problems and obstacle problems. In this case, two main difficulties should be overcome, one arises from the inherent nonlinearity of problems, the other stems from the non-consistency of the WOPSIP DG method. As a result, since we have to bound the error terms arising from the con-consistency (see (24) and (25) in Theorem 2.4), our error analysis is more involved than the proof in [15,16].

The rest of our paper is organized as follows. An optimal a priori error estimate of the WOPSIP method for the Signorini problem is provided in Section 2. In Section 3, the WOPSIP method for the obstacle problem is proposed and analyzed. Finally, in Section 4, some numerical experiments supporting our theoretical analysis are presented.

2. The WOPSIP method for Signorini problems

2.1. Problem set-up and notation

This section is devoted to the development of the WOPSIP method for the following Signorini problem:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \partial_{\mathbf{n}} u &= 0 \quad \text{on } \Gamma_N, \\ u \geq 0,\ \partial_{\mathbf{n}} u \geq 0,\ u\partial_{\mathbf{n}} u &= 0 \quad \text{on } \Gamma_C, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ and $f \in L^2(\Omega)$. $\partial_{\mathbf{n}} u$ is the outward normal derivative of u along the boundary, i.e., $\partial_{\mathbf{n}} u = \nabla u \cdot \mathbf{n}$.

Let us first introduce some notation. For a bounded domain $\mathcal{D}$ in $\mathbb{R}^2$, we denote by $H^s(\mathcal{D})$ the standard Sobolev space of functions with regularity exponent $s \geq 0$, associated with norm $\|\cdot\|_{s,\mathcal{D}}$ and seminorm $|\cdot|_{s,\mathcal{D}}$. When $s = 0$, $H^0(\mathcal{D})$ can be written by $L^2(\mathcal{D})$. Let

$$V = \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \right\},$$

and

$$K = \left\{ v \in V : v \geq 0 \text{ on } \Gamma_C \right\}. \tag{2}$$

Then the weak formulation of the Signorini problem (1) is to find $u \in K$ such that

$$\int_\Omega \nabla u \cdot \nabla (v - u) dx \geq \int_\Omega f(v - u) dx \quad \forall v \in K.$$

Let $\mathcal{T}_h$ be a shape-regular decomposition of Ω into triangles $\{T\}$, the diameter of T is denoted by h_T and $h = \max_{T \in \mathcal{T}_h} h_T$. We denote by $\mathcal{E}_h^I$ the set of interior edges of elements in $\mathcal{T}_h$. We assume that Γ_D, Γ_N and Γ_C are aligned with the triangulations $\mathcal{T}_h$, i.e., the end points of Γ_D and Γ_C coincide with the vertices of some elements. The subset of edges on Γ_D, Γ_N and Γ_C is denoted by $\mathcal{E}_h^D$, $\mathcal{E}_h^N$, and $\mathcal{E}_h^C$, respectively. Then the set of all edges $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^D \cup \mathcal{E}_h^N \cup \mathcal{E}_h^C$. The length of any edge $e \in \mathcal{E}_h$ is denoted by h_e. In addition, for each edge $e \in \mathcal{E}_h$ we associate with a fixed unit normal $\mathbf{n}$, such that for edges on the boundary Γ, $\mathbf{n}$ is the exterior unit normal.

Let e be an interior edge in $\mathcal{E}_h^I$ shared by two adjacent elements T_1 and T_2. For a scalar-valued piecewise smooth function φ, with $\varphi^i = \varphi|_{T_i}$, we define the following jump and average as follows:

$$[\![\varphi]\!] = \varphi^1 - \varphi^2 \quad \text{on } e \in \mathcal{E}_h^I,$$

$$\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2) \quad \text{on } e \in \mathcal{E}_h^I.$$

For a boundary edge e on Γ, we define

$$[\![\varphi]\!] = \varphi, \quad \text{and} \quad \{\varphi\} = \varphi.$$

The discontinuous P_1 finite element space corresponding to $\mathcal{T}_h$ is defined by

$$V_h = \left\{ v \in L^2(\Omega) : v|_T \in P_1(T),\ \forall T \in \mathcal{T}_h \right\}.$$

Moreover, we introduce the following convex subset of V_h to approximate the set K in (2):

$$K_h = \left\{ v_h \in V_h : v_h(x) \geq 0 \text{ at all nodes on } \overline{\Gamma_C} \right\}.$$

For any $v_h \in K_h$, $v_h \geq 0$ at all nodes on $\overline{\Gamma_C}$ yields $v_h \geq 0$ on $\overline{\Gamma_C}$ since we use the linear finite element.