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Global bifurcation and positive solution for a class of fully nonlinear problems[☆]

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ABSTRACT

In this paper, we study global bifurcation phenomena for the following Kirchhoff type problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u(x)|^2\,dx\right)\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where M is a continuous function. Under some natural hypotheses, we show that $(\lambda_1(a)M(0), 0)$ is a bifurcation point and there is a global continuum $\mathcal{C}$ emanating from $(\lambda_1(a)M(0), 0)$, where $\lambda_1(a)$ denotes the first eigenvalue of the above problem with $f(x,s) = a(x)s$. As an application of the above result, we study the existence of positive solution for this problem with asymptotically linear nonlinearity.

1. Introduction

Consider the following Kirchhoff type problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u(x)|^2\,dx\right)\Delta u = \lambda a(x)u(x) + g(x,u,\lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, M is a continuous function on $\mathbb{R}^+$, $a \in L^\infty(\Omega)$ with $a \not\equiv 0$, $\lambda > 0$ is a parameter, $g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the carathéodory condition in the first two variables and

$$\lim_{s\to 0} \frac{g(x,s,\lambda)}{s} = 0 \tag{1.2}$$

uniformly for a.e. $x \in \Omega$ and λ on bounded sets. Moreover, we also assume that g satisfies the growth restriction

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(G) There exist $c > 0$ and $p \in (1, 2^*)$ such that

$$|g(x, s, \lambda)| \le c\left(1 + |s|^{p-1}\right)$$

for a.e. $x \in \Omega$ and λ on bounded sets, where

$$2^* = \begin{cases} \dfrac{2N}{N-2}, & \text{if } N > 2, \\ +\infty, & \text{if } N \le 2. \end{cases}$$

The problem (1.1) is nonlocal as the appearance of the term $\int_\Omega |\nabla u(x)|^2\,dx$ which implies that it is not a pointwise identity. This causes some mathematical difficulties which make the study of problem (1.1) particularly interesting. Moreover, problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string [1]. After the famous paper by Lions [2], this type of problems has been the subject of numerous studies, and some important and interesting results have been obtained, for example, see [3–6]. Recently, there are many mathematicians studying this kind of problems by variational method, see [7–13] and the references therein. We refer to [14–20] for Kirchhoff models with critical exponents. For evolution problems, we refer to [21–23] and the references therein.

To the best of our knowledge, there are few papers that studied Kirchhoff type problems using the bifurcation theory, see for example [24,25]. The first aim of this paper is to study global bifurcation phenomena for problem (1.1). Let $\lambda_1(a)$ denote the first eigenvalue of the following problem

$$\begin{cases} -\Delta u = \lambda a(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

It is well known that $\lambda_1(a)$ is simple, isolated and is the unique principle eigenvalue of problem (1.3). Now, we make the following assumptions on M.

(M_0) M is a continuous function on $\mathbb{R}^+$ such that for some $m_0 > 0$, we have

$$M(t) \ge m_0, \quad \text{for all } t \in \mathbb{R}^+;$$

(M_1) there exists $m_1 > 0$, such that $\lim_{t\to+\infty} M(t) = m_1$.

The hypothesis (M_0) shows that our problem is non-degenerate. In [14,16] the so-called "degenerate" case is covered (see also [22,23,20]), that is the main Kirchhoff non-negative function M could be zero at 0.

Our first main result is the following theorem.

Theorem 1.1. *Assume that (1.2), (G) and (M_0) hold. Then $(\lambda_1(a)M(0), 0)$ is a bifurcation point of problem (1.1) and the associated bifurcation continuum $\mathcal{C}$ in $\mathbb{R} \times H_0^1(\Omega)$, whose closure contains $(\lambda_1(a)M(0), 0)$, is either unbounded or contains a pair $(\mu M(0), 0)$, where μ is another eigenvalue of problem (1.3).*

On the basis of Theorem 1.1, the second aim of this paper is to determine the interval of λ, for which there exists a positive solution for the following Kirchhoff type problem

$$\begin{cases} -M\left(\displaystyle\int_\Omega |\nabla u(x)|^2\,dx\right)\Delta u = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $f \in C\left(\overline{\Omega} \times \mathbb{R}\right)$ satisfies that

(f_1) $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x, s)s > 0$ for $x \in \Omega$ and any $s > 0$;
(f_2) $\lim_{s\to0^+} \frac{f(x,s)}{s} = a(x)$, $\lim_{s\to+\infty} \frac{f(x,s)}{s} = c(x) \not\equiv 0$ uniformly in $x \in \Omega$, where $a(x), c(x)$ such that they are strict positive on some subset of positive measure in Ω and $\lambda_1(c)m_1 \ne \lambda_1(a)M(0)$.

The following theorem is our second main result.

Theorem 1.2. *Suppose that (M_0)–(M_1) and (f_1)–(f_2) hold, then for*

$$\lambda \in \left(\min\{\lambda_1(c)m_1, \lambda_1(a)M(0)\},\ \max\{\lambda_1(c)m_1, \lambda_1(a)M(0)\}\right),$$

problem (1.4) possesses at least one positive solution.

Remark 1.3. Note that the corresponding existence result of [7] is a corollary of Theorem 1.2. In fact, by the monotonicity of eigenvalue with respect to weight, we get $1 \in (\lambda_1(a)M(0), \lambda_1(c)m_1)$ under the assumptions of Theorem 1 in [7]. So problem (1.4) with $\lambda = 1$ possesses at least one positive solution. Clearly, our assumptions are weaker than corresponding ones of [7]. Therefore, we improve and extend the corresponding result of [7].

The rest of this paper is organized as follows. Sections 2 and 3 present the proofs of Theorems 1.1 and 1.2, respectively.