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On augmentation block triangular preconditioners for regularized saddle point problems

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ABSTRACT

Two augmentation block triangular preconditioners were introduced by Shen et al. (2012) for the regularized saddle point problem. However, the spectral analysis of the preconditioner based on the augmentation of the (2, 2) block was not throughly derived there. In this short paper, we give a detailed spectral analysis on this preconditioner. A numerical example is employed to validate the efficiency of the presented theoretical results.

We consider the following *nonsingular* and regularized saddle point problem

$$\mathcal{A}x \equiv \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite, and $B \in \mathbb{R}^{m\times n}$ with $\mathrm{rank}(B) = m$. A variety of applications and numerical solution methods of (1) have been comprehensively reviewed by Benzi, Golub, and Liesen [1].

As is well known, Krylov subspace methods play central roles for solving (1), but preconditioners are usually required. Recently, augmentation techniques have been used directly or indirectly to construct block preconditioners for (1); see, e.g., [2–14]. In particular, the following augmentation block triangular preconditioners were given in [14] for (1):

$$\mathcal{P}_V = \begin{pmatrix} A + B^T V^{-1} B & 2B^T \\ 0 & -(V + C) \end{pmatrix}, \qquad \mathcal{R}_W = \begin{pmatrix} W + A & 0 \\ 2B & -(C + BW^{-1}B^T) \end{pmatrix},$$

where $V \in \mathbb{R}^{m\times m}$ and $W \in \mathbb{R}^{n\times n}$ are symmetric positive definite weighted matrices.

The preconditioners $\mathcal{R}_W$ and $\mathcal{P}_V$ can be efficiently applied to some saddle point problems with special properties or structures. For instance, they are very competitive to the HSS preconditioners (see, e.g., [15–17]) and constraint preconditioners (see, e.g., [18–20]) for the models of image restoration problems and weighted least squares problems; see [14]. In the implementation of the preconditioned iteration, one application of $\mathcal{R}_W$ needs two solves with $W + A$ and $C + BW^{-1}B^T$. Since n is often much larger than m [1], the system with $C + BW^{-1}B^T$ often has a small scale, and can be relatively easy to be solved. For some saddle point problems with $n \gg m$, the preconditioner $\mathcal{R}_W$ can be more suitable than the preconditioner $\mathcal{P}_V$.

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The spectral properties of $\mathcal{P}_V^{-1}\mathcal{A}$ have been given in detail by [14, Theorems 2.1–2.3]. However, the spectral analysis on $\mathcal{R}_W$ was not completely given there. The aim of this paper is to present some spectral analysis on $\mathcal{R}_W$, which can contribute to analyze the convergence behaviors of the corresponding preconditioned Krylov subspace methods. A numerical example is included to illustrate the sharpness of these theoretical results.

Throughout the paper, we define

$$\bar{A} = W^{-\frac{1}{2}}AW^{-\frac{1}{2}}, \qquad \bar{B} = BW^{-\frac{1}{2}}, \qquad \widehat{B} = (C + \bar{B}\bar{B}^T)^{-\frac{1}{2}}\bar{B}, \tag{2}$$

$$\widehat{C} = (C + \bar{B}\bar{B}^T)^{-\frac{1}{2}}C(C + \bar{B}\bar{B}^T)^{-\frac{1}{2}}. \tag{3}$$

Let $E, F \in \mathbb{R}^{p\times p}$, $z \in \mathbb{C}$. Then the inequality $E \succ F$ (respectively, $E \succeq F$) means that $E - F$ is symmetric positive definite (respectively, semidefinite). We write $E \prec F$ (respectively, $E \preceq F$) if and only if $F \succ E$ (respectively, $F \succeq E$). We denote by $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\lambda_{\min}(E)$ and $\lambda_{\max}(E)$ the real part and the imaginary part of z, the minimum eigenvalue and the maximum eigenvalue of E with all eigenvalues being real, respectively.

We first consider the case $A \succeq W$ or $A \preceq W$.

Theorem 1. *Let $\bar{A}$ and $\widehat{B}$ be defined as in (2), and let*

$$\mu_1 = \lambda_{\min}(\bar{A}), \qquad \mu_n = \lambda_{\max}(\bar{A}), \qquad \nu_m = \lambda_{\max}(\widehat{B}\widehat{B}^T). \tag{4}$$

Then the preconditioned matrix $\mathcal{R}_W^{-1}\mathcal{A}$ is similar to $\mathcal{I}_{n+m} - \mathcal{M}$, where

$$\mathcal{M} = \begin{pmatrix} H & 0 \\ 2\widehat{B}(I_n + \bar{A})^{-1} - \widehat{B} & 0 \end{pmatrix}$$

with

$$H = (I_n + \bar{A})^{-1} + \widehat{B}^T\widehat{B}(\bar{A} - I_n)(I_n + \bar{A})^{-1}. \tag{5}$$

Consequently, the eigenvalues of $\mathcal{R}_W^{-1}\mathcal{A}$ are the eigenvalues of $I_n - H$ together with m eigenvalues equal to 1. Furthermore, the following hold:

(i) *If $A \succeq W$, then the eigenvalues different from 1 of $\mathcal{R}_W^{-1}\mathcal{A}$ are all real, and belong to*

$$\left[\Phi_1, \ \frac{\mu_n}{1+\mu_n}\right], \tag{6}$$

where

$$\Phi_1 = \begin{cases} \dfrac{\mu_n + \nu_m - \mu_n\nu_m}{1+\mu_n}, & \nu_m \in \left[\dfrac{1}{2}, \ 1\right] \\ \dfrac{\mu_1 + \nu_m - \mu_1\nu_m}{1+\mu_1}, & \nu_m \in \left[0, \ \dfrac{1}{2}\right]. \end{cases}$$

(ii) *If $A \preceq W$, then the eigenvalues different from 1 of $\mathcal{R}_W^{-1}\mathcal{A}$ are all real, and belong to*

$$\left[\frac{\mu_1}{1+\mu_1}, \ \Phi_2\right], \tag{7}$$

where

$$\Phi_2 = \begin{cases} \dfrac{\mu_1 + \nu_m - \mu_1\nu_m}{1+\mu_1}, & \nu_m \in \left[\dfrac{1}{2}, \ 1\right] \\ \dfrac{\mu_n + \nu_m - \mu_n\nu_m}{1+\mu_n}, & \nu_m \in \left[0, \ \dfrac{1}{2}\right]. \end{cases}$$

Proof. Let

$$\mathcal{D} = \begin{pmatrix} W & 0 \\ 0 & I_m \end{pmatrix}.$$

Then the preconditioned matrix $\mathcal{R}_W^{-1}\mathcal{A}$ is similar to

$$\begin{aligned} \mathcal{D}^{\frac{1}{2}}\mathcal{R}_W^{-1}\mathcal{A}\mathcal{D}^{-\frac{1}{2}} &= (\mathcal{D}^{-\frac{1}{2}}\mathcal{R}_W\mathcal{D}^{-\frac{1}{2}})^{-1}(\mathcal{D}^{-\frac{1}{2}}\mathcal{A}\mathcal{D}^{-\frac{1}{2}}) \\ &= \begin{pmatrix} I_n + \bar{A} & 0 \\ 2\bar{B} & -(C + \bar{B}\bar{B}^T) \end{pmatrix}^{-1} \begin{pmatrix} \bar{A} & \bar{B}^T \\ \bar{B} & -C \end{pmatrix} := \widetilde{\mathcal{R}}_W^{-1}\widetilde{\mathcal{A}}, \end{aligned} \tag{8}$$