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Analysis of least squares pseudo-spectral method for the interface problem of the Navier–Stokes equations

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ABSTRACT

The aim of this paper is to propose and analyze the first order system least squares method for the incompressible Navier–Stokes equation with discontinuous viscosity and singular force along the interface as the earlier work of the first author on Stokes interface problem (Hessari, 2014). Interface conditions are derived, and the Navier–Stokes equation transformed into a first order system of equations by introducing velocity gradient as a new variable. The least squares functional is defined based on L^2 norm applied to the first order system. Both discrete and continuous least squares functionals are put into the canonical form and the existence and uniqueness of branch of nonsingular solutions are shown. The spectral convergence of the proposed method is given. Numerical studies of the convergence are also provided.

1. Introduction

In this paper, we consider the Navier–Stokes equation with discontinuous viscosity and singular force, which in the steady case reads

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} + \mathbf{h}\delta_\Gamma & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ is velocity vector, p is pressure, and $\mathbf{f}$ is external force function. The domain Ω is an open bounded domain separated into two sub-domains Ω_1 and Ω_2, by curve Γ, such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \Gamma$. Here, Γ is referred to as *interface*. The boundary of Ω is denoted by $\partial\Omega$. Let $\partial\Omega_1 = \overline{\Omega_1} \cap \partial\Omega$ and $\partial\Omega_2 = \overline{\Omega_2} \cap \partial\Omega$. The function $\mathbf{h}$ is force density defined only on the interface Γ and δ_Γ is 2-dimensional delta function with the support along the interface Γ. We assume that the viscosity ν is piecewise constant, defined by

$$\nu(x,y) = \begin{cases} \nu_1, & \text{if } (x,y) \in \Omega_1, \\ \nu_2, & \text{if } (x,y) \in \Omega_2. \end{cases}$$

For uniqueness, we also impose zero mean pressure condition

$$\int_\Omega p\,dx = 0. \tag{1.2}$$

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The solution of this problem due to the presence of singular source term and discontinuous viscosity coefficient is nonsmooth across the interface. We call this as *Navier–Stokes interface problem*. Several methods such as finite difference and finite element method have been proposed for this problem. Peskin's immersed boundary model that was introduced to study the fluid dynamics of blood flow in the human heart [1] is one of the most successful Cartesian grid methods. The method has been developed and applied to many fluid and biological problems. The immersed boundary method [2–4] has been used for problems with non-smooth but continuous solution which is only first order accurate. The immersed interface method (IIM) which is second order accurate was developed by LeVeque and Li [5] for the elliptic interface problem, and then generalized to Stokes [6] and Navier–Stokes problem [7,8]. Li and Lai [8] proposed the immersed interface method for the incompressible Navier–Stokes equations with singular forces along the interface which is second order accurate for the velocity and nearly second order accurate for the pressure in the maximum norm. Lee and LeVeque [7] also presented the immersed interface method for the incompressible Navier–Stokes equations with second order accuracy, using finite volume method. The spectral collocation method also has been used to approximate the solution of interface problems. Shin and Jung [9] presented the spectral collocation method for one dimensional interface problems. Hessari and Shin [10] have developed an algorithm to approximate the solutions of second order elliptic interface problems. In the immersed interface method several jump conditions are used in order to get accurate numerical solution. These interface conditions include coupled interface condition for pressure and velocity, and zero's, first and second order derivatives of velocity and pressure. The numerous number of interface conditions as well as being coupled, make it difficult to use first order system least squares method for Navier–Stokes interface problem. In this paper, we generalize the methodology given in [11] for Stokes interface problem using pseudo-spectral least squares principle. The accuracy of spectral methods which employ the global polynomial for discretization, and least squares method with its advantages are combined to approximate the solution of Navier–Stokes interface problem (1.1). Among the advantages of least squares method is that the choice of approximation spaces for velocity and pressure are not subject to LBB compatibility condition and one can use the equal order interpolation polynomials to approximate all variables. In addition, the algebraic system is always symmetric and positive definite and can be preconditioned. We note that in all the above mentioned works, some algorithm is given to solve Navier–Stokes interface problems numerically. In this work, however, we provide analysis of existence and uniqueness of branch of nonsingular solution and its convergence.

To employ least squares pseudo-spectral method for the Navier–Stokes interface problem, we extend the methodology presented in [11]. To do this, we first apply finite element argument to the problem (1.1), to derive interface conditions and Navier–Stokes equations in each sub-domain separately. The Navier–Stokes equation in each sub-domain is transformed into the first order system and then extended by some identities. The least squares functional is defined by summing up the squared L^2-norm of residual of extended first order system and squared L^2-norm of coupled jump condition for pressure and velocity. The jump condition for velocity (continuity of velocity across interface) is imposed into the velocity solution space. Actually, velocity solution space includes essential boundary condition and velocity jump condition. We also note that the analysis given here is for arbitrary domain Ω, however, the numerical experiment is done only for rectangle domain with straight line interface. In the case of curved interface, one can use the Gordon–Hall transformation (see [12,13,10,14] for more details).

The contents of this paper are organized as follows. In the following section, we provide preliminaries which is needed in subsequent sections. Interface conditions are derived in Section 3. The Navier–Stokes interface problem is transformed into a system of first order equations in 4. The analysis of least squares pseudo-spectral method as well as its convergence is provided in Section 5. Section 6 gives implementation with a numerical test. We finalized the paper with conclusion remarks in Section 7.

2. Preliminaries

In this section, some preliminaries, definitions and notations are given. The standard notations and definitions are used for the weighted Sobolev spaces $H^s_\omega(\mathcal{D})$ equipped with weighted inner product $(\cdot,\cdot)_{s,\omega}$ and corresponding weighted norms $\|\cdot\|_{s,\omega}$, $s \geq 0$, where ω is the Legendre weight function $\omega(x, y) = 1$, and $\mathcal{D} = [-1, 1]^2$. The space $H^0_\omega(\mathcal{D})$ coincides with $L^2_\omega(\mathcal{D})$, in which case the norm and inner product will be denoted by $\|\cdot\|_\omega$ and $(\cdot,\cdot)_\omega$, respectively. For simplicity, we write the notations without the subscript ω. Let $L^2_0(\mathcal{D})$ be the subspace of $L^2(\mathcal{D})$ whose functions have average zero, i.e., $\int_{\mathcal{D}} p\, w\, dx = 0$.

Let $\mathcal{P}_N$ be the space of all polynomials of degree less than or equal to N, and $\{\xi_i\}_{i=0}^N$ be the Legendre–Gauss Lobatto (LGL) points on $[-1, 1]$ such that

$$-1 := \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N := 1.$$

Here $\{\xi_i\}_{i=0}^N$ are the zeros of $(1 - t^2)L'_N(t)$, where L_N is the N-th Legendre polynomial and the corresponding quadrature weights $\{\omega_i\}_{i=0}^N$ are given by

$$\begin{cases} \omega_j = \dfrac{2}{N(N+1)} \dfrac{1}{[L_N(\xi_j)]^2}, & 1 \leq j \leq N-1, \\ \omega_0 = \omega_N = \dfrac{2}{N(N+1)}. \end{cases} \tag{2.1}$$