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Symmetric and non-symmetric waves in the osmosis K(2, 2) equation

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ABSTRACT

We give an improved qualitative method to solve the osmosis K(2, 2) equation. This method combines several characteristics of other methods. Using this method, the existence of symmetric and non-symmetric wave solutions of the osmosis K(2, 2) equation is studied. Besides abundant symmetric forms such as smooth wave solutions, peaked waves, cusped waves, looped waves, stumpons and fractal-like waves, this equation also admits non-symmetric ones including breaking kink wave solutions, breaking anti-kink wave solutions and rampons. As regards this equation most of those solutions, either symmetric or non-symmetric solutions, have not appeared in the literature. We also study the limiting behavior of all periodic solutions as the parameters tend to some special values.

1. Introduction

As we know, convection effects steepent of the wave equation and dispersion can cause the wave equation to spread. The solution is the result of the interaction between the convection and dispersion.

The well-known Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

has a convection term uu_x and a linear dispersion term u_{xxx}. The interaction of these two terms generates exactly smooth solitary wave solutions [1,2].

To study the role of nonlinear dispersion in the formation of patterns in a liquid drop, Rosenau and Hyman [3] obtained and studied the K(2, 2) equation

$$u_t + (u^2)_x + (u^2)_{xxx} = 0.$$

The interaction of nonlinear dispersion with nonlinear convection can compactify solitary waves and generate compactons: solitons with finite wavelength or robust soliton-like solutions characterized by the absence of infinite wings.

Recently, by the bifurcation method of phase plane, Tian et al. investigated the osmosis K(2, 2) equation [4,5]

$$u_t + (u^2)_x - (u^2)_{xxx} = 0, \tag{1.1}$$

where the negative dispersive term denotes the contracting dispersion. In this case, they found that Eq. (1.1) yielded more solutions such as peaked waves, periodic cusped waves and smooth waves. Those solutions can be considered as the interaction of nonlinear contracting dispersion with nonlinear convection. For more research on the K(2, 2) equation, one can turn to references herein [6–8].

In this paper, we are interested in whether this kind of interaction could generate many other structures otherwise unattainable.

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It is worth noting here that the famous Camassa–Holm (CH) equation [9]

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

has similar structure with the osmosis K(2, 2) equation. In fact, we find that the CH equation contains exactly the same nonlinear terms as in Eq. (1.1), and the only difference is the relative coefficients. However, abundant solutions of the CH equation have been given by different methods: The peaked solitary waves were obtained by the bifurcation method of phase plane [10]. The analytic expressions of peaked periodic wave solutions were given by using bifurcation method of planar dynamical systems and the convergence of the peaked periodic wave solutions was also proven [11]. The peaked waves and cusped waves were found by the method of Jacobian elliptic function [12]. The cusped wave solutions in parametric form were obtained by the factorization method. In [13], the authors studied the dispersionless limit in which the cusped wave converges to an anti-peaked wave.

Unfortunately, the above mentioned methods are limited in that the non-smooth wave solutions such as peaked and cusped forms have not been checked in a weak solution point of view. As a result more new non-smooth solutions could not been given. Lenells made great contribution in solving this problem [14]. He gave a definition of weak solutions under which the abundant travelling wave solutions of the CH equation have been determined, including peaked waves, cusped waves and some composite waves. However, some interesting solutions such as looped forms, breaking kink forms and other non-symmetric forms were not considered in his method. The explicit solutions were given only for the peaked solitary waves and periodic peaked waves. Moreover, the limiting behavior of solutions had not been studied yet.

Clearly, the results in [4,5] for the Eq. (1.1) are incomplete. In this paper, we will give an improved method combining some characteristics of the above mentioned methods to further study the existence of travelling wave solutions of (1.1) in every parameter region of the parameter space. Our method has the following aspects: (i) More solutions forms are considered and their explicit expressions are given. Among them, looped waves, stumpons and fractal-like waves and rampons are new solutions of Eq. (1.1). (ii) The parameter space is divided in further details. (iii) Some strange composite wave solutions in the weak solutions sense are got. (iv) The limiting behavior of all periodic solutions is given.

This paper is organized as follows. In Section 2, we give the definition of a weak solution and the theorem of the classification of travelling waves in Eq. (1.1). In Section 3, the proof of Theorem 1 is given. Last section is the conclusion.

2. Main results

In this section we will give the classification of travelling wave solutions of Eq. (1.1), which is stated in Theorem 1.

For a travelling wave $u(x, t) = \phi(x - ct)$, Eq. (1.1) takes the form

$$-c\phi_x + (\phi^2)_x - (\phi^2)_{xxx} = 0, \tag{2.1}$$

where c is the wave speed. As we know, if $\phi(x - ct)$ is a travelling wave solution of Eq. (1.1), we can also obtain another travelling wave solution $-\phi[-(x - ct)]$ of (1.1) with c replaced by $-c$. Therefore, we will only consider travelling wave solutions with a positive speed $c > 0$.

By integrating with respect to x, (2.1) is equivalent to the following integrated form

$$(\phi^2)_{xx} = \phi^2 - c\phi + \alpha, \tag{2.2}$$

where α is an integral constant. Eq. (2.2) makes sense for all $\phi \in H^1_{loc}(\mathbb{R})$. The following definition is therefore natural.

Definition 1. A function $\phi \in H^1_{loc}(\mathbb{R})$ is a travelling wave solution of Eq. (1.1) if ϕ satisfies (2.2) in distribution sense for some $\alpha \in \mathbb{R}$.

By Definition 1 and Lemmas 4 and 5 in [14], we can give the following definition of weak travelling wave solutions.

Definition 2. Any bounded function ϕ belongs to $H^1_{loc}(\mathbb{R})$ and is a travelling wave solution of Eq. (1.1) with speed c if and only if satisfying the following two statements:

(A) There are disjoint open intervals E_i, $i \geq 1$, and a closed set C such that $\mathbb{R} \setminus C = \bigcup_{i=1}^{\infty} E_i$, $\phi \in C^\infty(E_i)$ for $i \geq 1$, $\phi(x) \neq 0$ for $x \in \bigcup_{i=1}^{\infty} E_i$ and $\phi(x) = 0$ for $x \in C$.

(B) There is an $\alpha \in \mathbb{R}$ such that

(i) For each $\alpha \in \mathbb{R}$, there exists $\beta \in \mathbb{R}$ such that

$$\phi_x^2 = F(\phi), \quad x \in E_i \tag{2.3a}$$

where

$$F(\phi) = \frac{\frac{1}{2}\phi^2(\phi^2 - \frac{4c}{3}\phi + 2\alpha) + \beta}{2\phi^2} \tag{2.3b}$$

and $\phi \to 0$, at any finite endpoint of E_i.

(ii) If C has strictly positive Lebesgue measure $\mu(C) > 0$, we have $\alpha = 0$.

Remark. Here we show how (2.3) are derived. Multiplying both sides in Eq. (2.2) by $(\phi^2)_x$ and integrating once, we get (2.3).