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Integral boundary value problems with causal operators

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ABSTRACT

This paper considers the existence of solutions for a class of integral boundary value problems with causal operators. We build a new comparison theorem. By utilizing the monotone iterative technique and the method of lower and upper solutions, we formulate sufficient conditions under which such problems have extremal or quasisolutions in a corresponding sector.

1. Introduction

In this paper, we deal with the following differential equation with a causal operator:

$$\begin{cases} y'(t) = (Qy)(t), \quad t \neq t_k,\ t \in J = [0, T], \\ \Delta y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \dots, m, \\ y(0) = \lambda_1 y(\tau) + \lambda_2 \displaystyle\int_0^T \mu(t, y(t))\mathrm{d}t + c, \end{cases} \tag{1.1}$$

where $t \in J = [0, T]$ $(T > 0)$, $E = C[J, R]$, and $Q \in C[E, E]$ is a causal operator, $\tau \in (0, T]$, $\mu \in C(J \times R, R)$, $\lambda_1, \lambda_2, c \in R$, $0 < t_1 < t_2 < \cdots < t_m < T$, $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, $t_{m+1} = T$, $I_k \in C(R, R)$. $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, $k = 1, 2, \dots, m$.

If $\lambda_1 = 1, \lambda_2 = c = 0$ and $\tau = T$, then we have the periodic boundary condition, if $\lambda_1 = -1, \lambda_2 = c = 0$ and $\tau = T$, then we have the anti-periodic boundary condition, and if $\lambda_1 = \lambda_2 = 0$, we have an initial condition as special cases of the boundary condition in (1.1).

Remark 1.1. We note that the boundary value condition in (1.1) would change if τ changes from zero to any positive constant in interval $(0, T]$, even that $\tau = t_k$, $k = 1, 2, \dots, m$. That is, our boundary value condition has a very general form.

Some authors have focused their interest on differential equations with causal operators recently; see [1–4]. A causal operator is a non-anticipative operator. Its theory has the powerful quality of unifying ordinary differential equations, integro-differential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral functional equations, to name a few. Impulsive differential equations are a class important models, which describe many evolution processes that abruptly change their state at a certain moment (see [5–9]) and have been studied in depth by some authors in recent years.

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To obtain existence results for differential equations, someone used the monotone iterative method (see [10,11]). There is a vast amount of literature devoted to the applications of this method to differential equations with initial and boundary conditions (see [12–15]). We also apply this technique to problem (1.1).

In this paper, we extend the notion of causal operators to the integral boundary value problems. First, some comparison principles and the existence and uniqueness of the solutions for the first order linear differential equations with linear boundary conditions are presented. Next, by utilizing the monotone iterative technique and the method of lower and upper solutions, we establish the existence of extremal solutions of the problem (1.1). At last, using the notion of coupled lower and upper solutions, we prove the existence of the coupled quasisolutions of the problem (1.1). An example is added to verify assumptions and theoretical results.

2. Preliminaries

In this section, we present some definitions and lemmas which help to prove our main results.

Let $J^- = J \setminus \{t_1, t_2, \ldots, t_m\}$, $PC(J, R) = \{y : J \to R \mid y(t)$ is continuous everywhere except for some t_k at which $y(t_k^-) = y(t_k),\ k = 1, 2, \ldots, m\}$. $PC^1(J, R) = \{y \in PC(J, R) \mid y'$ is continuous on J^-, where $y'(0^+),\ y'(T^-),\ y'(t_k^+)$ and $y'(t_k^-)$ exist, $k = 1, 2, \ldots, m\}$. Let $E_0 = PC(J, R)$ with norm

$$\|y\|_{E_0} = \sup_{t\in J} |y(t)|,$$

then E_0 is a Banach space, and let $\Omega = PC^1(J, R)$. A function $y \in \Omega$ is called a solution of (1.1) if it satisfies (1.1).

Definition 2.1. Suppose that $Q \in C[E, E]$, then Q is said to be a causal map or a nonanticipative map if $u(s) = v(s)$ for $0 \le s \le t \le T$ when $u, v \in E$, then

$$(Qu)(s) = (Qv)(s), \quad 0 \le s \le t.$$

The following comparison results and lemmas play an important role in this paper.

Lemma 2.1 (*Comparison Theorem*). *Let $m \in \Omega$ satisfies*

$$\begin{cases} m'(t) \le -M(t)m(t) - (\zeta m)(t), & t \ne t_k,\ t \in J, \\ \Delta m(t_k) \le -L_k m(t_k), & k = 1, 2, \ldots, m, \\ m(0) \le 0 \end{cases} \tag{2.1}$$

where $0 \le L_k < 1$, $k = 1, 2, \ldots, m$, and $M(t)$ is a non-negative integrable function, $\zeta \in C[E, E]$ is a positive linear operator, that is, $\zeta m \ge 0$ whenever $m \ge 0$. In addition, we assume that

$$\int_0^T [M(t) + \zeta(t)]\mathrm{d}t + \sum_{k=1}^{m} L_k \le 1, \tag{2.2}$$

then $m(t) \le 0$, $\forall t \in J$.

Proof. Assume that $m(t) \le 0$, $\forall t \in J$ is not true, then there exists a $t^* \in (0, T]$ such that $m(t^*) > 0$. Let $\min_{0\le t\le t^*} m(t) = -b$, then $b \ge 0$.

Case 1: if $b = 0$, then $m(t) \ge 0$, $\forall t \in [0, t^*]$. Thus, by (2.1), we have $m'(t) \le 0$, $\forall t \in [0, t^*]$, and $\Delta m(t_k) \le -L_k m(t_k) \le 0$, $0 < t_k < t^*$, hence $m(t)$ is non-increasing in $[0, t^*]$. So, we have $m(t^*) \le m(0) \le 0$, which contradicts $m(t^*) > 0$.

Case 2: if $b > 0$, then there exists a $t_* \in J_j$ $(j \le m)$ such that $m(t_*) = -b < 0$, or $m(t_*^+) = -b$. We only consider $m(t_*) = -b$, as for the case $m(t_*^+) = -b$, the proof is similar. Now from (2.1), we have

$$\begin{aligned} 0 < m(t^*) &= m(t_*) + \int_{t_*}^{t^*} m'(t)\mathrm{d}t + \sum_{t_*<t_k<t^*} \Delta m(t_k) \\ &\le -b + b\int_0^T [M(t) + \zeta(t)]\mathrm{d}t + b\sum_{k=1}^{m} L_k \\ &= b\left\{\int_0^T [M(t) + \zeta(t)]\mathrm{d}t + \sum_{k=1}^{m} L_k - 1\right\}. \end{aligned}$$

Then, we get

$$\int_0^T [M(t) + \zeta(t)]\mathrm{d}t + \sum_{k=1}^{m} L_k > 1,$$

which contradicts (2.2). Hence we have $m(t) \le 0$, $\forall t \in J$. The proof is complete. □