Computers and Mathematics with Applications 59 (2010) 2827–2835

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Nordhaus–Gaddum relations for proximity and remoteness in graphs

M. Aouchiche *, P. Hansen

GERAD and HEC Montreal, Montreal, QC, Canada

ARTICLE INFO

Article history:
Received 30 June 2008
Received in revised form 4 August 2009
Accepted 13 January 2010

Keywords:
Nordhaus–Gaddum
Proximity
Remoteness
Extremal graph

ABSTRACT

The transmission of a vertex in a connected graph is the sum of all distances from that vertex to the others. It is said to be normalized if divided by $n-1$, where n denotes the order of the graph. The proximity of a graph is the minimum normalized transmission, while the remoteness is the maximum normalized transmission. In this paper, we give Nordhaus–Gaddum-type inequalities for proximity and remoteness in graphs. The extremal graphs are also characterized for each case.

© 2010 Elsevier Ltd. All rights reserved.

1. Introduction

In this paper, $G=(V,E)$ denotes a simple connected graph, with vertex set V and edge set E, on $n=|V|$ vertices and $m=|E|$ edges. The degree of a vertex $v \in V$ is denoted by $d_G(v)$, or $d(v)$ when no confusion is possible. The minimum, average and maximum degrees are denoted by δ, $\overline{d}$ and Δ, respectively. The distance between two vertices u and v in G, denoted by $d(u,v)$, is the length of a shortest path between u and v. The average distance between all pairs of vertices in G is denoted by $\bar{l}$. The eccentricity $e(v)$ of a vertex v in G is the largest distance from v to another vertex of G. The minimum eccentricity in G, denoted by r, is the radius of G. The maximum eccentricity of G, denoted by D, is the diameter of G. The average eccentricity of G is denoted by ecc. That is,

$$r=\min_{v\in V} e(v), \qquad D=\max_{v\in V} e(v) \quad \text{and} \quad ecc=\frac{1}{n}\sum_{v\in V} e(v).$$

It is trivial that $r \le ecc \le D$ and $\bar{l} \le ecc$. The transmission $t(v)$ of a vertex v is the sum of the distances from v to all other vertices in G. It is said to be normalized, and then denoted $\tilde{t}(v)$, when divided by $n-1$. The proximity π and remoteness ρ [1,2] of G are, respectively, the minimum and the maximum normalized transmission in G. That is,

$$\pi=\min_{v\in V} \tilde{t}(v) \quad \text{and} \quad \rho=\max_{v\in V} \tilde{t}(v).$$

Note that, by definition,

$$\pi \le r \le ecc \le D, \qquad \pi \le \bar{l} \le \rho \le D \quad \text{and} \quad \bar{l}=\frac{1}{n}\sum_{v\in V} \tilde{t}(v).$$

Thus normalizing $t(v)$ helps in comparing graph invariants. Sharp bounds, proved in [3], on the proximity and the remoteness of a graph G on n vertices are

* Corresponding author.
E-mail addresses: mustapha.aouchiche@gerad.ca (M. Aouchiche), pierre.hansen@gerad.ca (P. Hansen).

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.02.001

$$1 \le \pi \le \begin{cases} \dfrac{n+1}{4} & \text{if } n \text{ is odd,} \\ \dfrac{n}{4} + \dfrac{n}{4(n-1)} & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad 1 \le \rho \le \frac{n}{2}.$$

The lower bound on π is reached if and only if G contains a dominating vertex, i.e. $\Delta = n - 1$; the upper bound on π is attained if and only if G is either the cycle C_n or the path P_n; the lower bound on ρ is reached if and only if G is the complete graph K_n; the upper bound on ρ is attained if and only if G is the path P_n.

Let G be a graph and $\overline{G}$ its complement. If I is an invariant of G, we denote by $\bar{I}$ the same invariant but in $\overline{G}$. Nordhaus–Gaddum relations for the graph invariant I are inequalities of the following form:

$$L_1(n) \le I + \bar{I} \le U_1(n) \quad \text{and} \quad L_2(n) \le I \cdot \bar{I} \le U_2(n),$$

where $L_1(n)$ and $L_2(n)$ are lower bounding functions of the order n, and $U_1(n)$ and $U_2(n)$ upper bounding functions of the order n. Note that sometimes, in addition to the order n, other graph invariants are used in the bounds. These types of relation are named after Nordhaus and Gaddum [4], who were the first authors to give such relations, namely

$$2\sqrt{n} \le \chi + \overline{\chi} \le n + 1 \quad \text{and} \quad n \le \chi \cdot \overline{\chi} \le \left(\frac{n+1}{2}\right)^2, \tag{1}$$

where χ is the chromatic number of a graph. Finck [5] characterized the extremal graphs for the inequalities in (1). Since then many graph theorists have been interested in finding such relations for various graph invariants. See [6] for a review of early results of Nordhaus–Gaddum type. A variety of more recent papers devoted to such results can be found in the graph theory literature, e.g. [7–14].

In order to get conjectures on the bounds in the Nordhaus–Gaddum-type inequalities for proximity and remoteness of a graph and its complement, we used the AutoGraphiX 2 (AGX 2, for short) system [15–18]. This "discovery system" is described, together with its results, in a series of papers under the common title "Variable Neighborhood Search for Extremal Graphs"; see [16] for references. It is based on the following observation: a large variety of problems in extremal graph theory can be viewed as parametric combinatorial optimization ones defined on the family of all graphs (or some restriction thereof) and solved by a generic heuristic. The parameter is usually the order n of the graphs considered (sometimes the order n and the size m or another graph invariant). The heuristic fits in the Variable Neighborhood Search metaheuristic framework [17,19,20]. Presumably extremal graphs are found by performing a series of local changes (removal, addition or rotation of an edge, etc.) until a local optimum is reached, then applying increasingly large perturbations, followed by new descents; if a graph better than the incumbent one is found, the search is recentered there. After the parametric family of extremal graphs has been found, relationships between graph invariants may be deduced from them using various data mining techniques [18]. These include (i) a numerical method based on Principal Component Analysis which yields a basis of affine relations between the graph invariants considered; (ii) a geometric method which uses a gift-wrapping algorithm to find the convex hull of extremal graphs viewed as points in the invariants space; facets of this convex hull give inequality relations; (iii) an algebraic method which recognizes families of graphs then exploits a database of formulae giving expressions of invariants as functions of n on these families; substitution then leads to linear or nonlinear conjectures.

2. Proximity

In this section, we prove results of Nordhaus–Gaddum type for proximity in a graph and its complement. First, we prove the lower and upper bounds, and characterize the associated extremal graphs, on the sum $\pi + \overline{\pi}$.

Theorem 1. *For any connected graph G on $n \ge 5$ vertices for which $\overline{G}$ is connected,*

$$\frac{2n}{n-1} \le \pi + \overline{\pi} \le \begin{cases} \dfrac{n+1}{4} + \dfrac{n+1}{n-1} & \text{if } n \text{ is odd,} \\ \dfrac{n}{4} + \dfrac{n}{4(n-1)} + \dfrac{n+1}{n-1} & \text{if } n \text{ is even.} \end{cases}$$

The lower bound is attained if and only if $\Delta(G) = \Delta(\overline{G}) = n - 2$. The upper bound is attained if and only if either G or $\overline{G}$ is the cycle C_n.

Proof.

Lower bound: Note that, if G is a connected graph such that $\overline{G}$ is also connected, then $\Delta \le n - 2$. So

$$\pi \ge \frac{\Delta + 2(n - \Delta - 1)}{n-1} = \frac{2n - \Delta - 2}{n-1} \ge \frac{(n-2)+2}{n-1} = \frac{n}{n-1}. \tag{2}$$

Thus, the lower bound follows. Moreover, equality in (2) holds if and only if $\Delta = n - 2$. Then, the lower bound is attained if and only if $\Delta = n - 2$ and $\overline{\Delta} = n - 2$ (or equivalently $\Delta = n - 2$ and $\delta = 1$).

Download English Version:

https://daneshyari.com/en/article/470991

Download Persian Version:

https://daneshyari.com/article/470991

Daneshyari.com