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Multiple solutions for fractional differential equations with nonlinear boundary conditions☆

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ABSTRACT

In this paper, we study certain fractional differential equations with nonlinear boundary conditions. By means of the Amann theorem and the method of upper and lower solutions, some new results on the multiple solutions are obtained.

1. Introduction

In this paper, we are concerned with the existence of multiple solutions for the fractional differential equations with nonlinear boundary conditions

$$\begin{cases} {}^C D^q x(t) = f(t, x(t), x'(t)), \quad t \in (0,1), \\ g_0(x(0), x'(0)) = 0, \\ g_1(x(1), x'(1)) = 0, \\ x''(0) = x'''(0) = \cdots = x^{(n-1)}(0) = 0, \end{cases} \tag{1.1}$$

where ${}^C D^q$ is the standard Caputo derivative, $n > 2$ is an integer, $q \in (n-1, n]$. $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and g_0, g_1 are given nonlinear functions.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [1–3] and the references therein.

Recently, there have been some papers which deal with the existence of the solutions of initial value problems or linear boundary value problems for fractional differential equations. In [4], the basic theories for the fractional calculus and the fractional differential equations are discussed. In [5–7], the basic theory for the initial value problems for fractional differential equations or fractional functional differential equations involving Riemann–Liouville differential operators are discussed. The results on general existence and uniqueness are proved by means of the monotone iterative technique and the method of upper and lower solutions.

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In [8], by using some fixed-point theorems on a cone, Bai investigates the existence and multiplicity of positive solutions for nonlinear fractional differential equations with linear boundary conditions

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$ is a real number, D^{α} is the standard Riemann–Liouville derivative.

In [9], the authors study the nonlinear fractional differential equation with linear boundary conditions

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t)), \quad t \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

where $3 < \alpha \leq 4$ is a real number, D^{α} is the standard Riemann–Liouville derivative. Some multiple positive solutions for singular and nonsingular boundary value problems are given.

However, research on the multiple solutions of the fractional differential equations with nonlinear boundary conditions is proceeding very slowly. In this paper, we focus on the multiple solutions for fractional differential equations with nonlinear boundary conditions. By means of the famous Amann theorem and the lower and upper solutions method, we obtain a new result on the existence of at least three distinct solutions under certain conditions.

2. Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

The definitions concerning the fractional integral and the Caputo fractional derivative can be found in the literature [4].

Lemma 2.1. *Suppose that $y \in C[0, 1]$ and $m_i, n_i \in \mathbb{R}, i = 1, 2$, with $m_1(n_1 + n_2) - m_2 n_1 \neq 0$. Then the following linear boundary value problem:*

$$\begin{cases} {}^{C}D^{q}x(t) = y(t), \quad t \in (0, 1), \\ m_1 x(0) + m_2 x'(0) = \lambda_0, \\ n_1 x(1) + n_2 x'(1) = \lambda_1, \\ x''(0) = x'''(0) = \cdots = x^{(n-1)}(0) = 0 \end{cases} \tag{2.1}$$

is equivalent to the following fractional integral:

$$\begin{aligned} x(t) &= \rho\left(\lambda_0\big(n_2 + n_1(1-t)\big) - \lambda_1(m_2 - m_1 t) + \frac{m_2 - m_1 t}{\Gamma(q)} \int_0^1 \big(n_1(1-s) + n_2(q-1)\big)(1-s)^{q-2} y(s)\mathrm{d}s\right) \\ &\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\mathrm{d}s, \end{aligned} \tag{2.2}$$

where $\rho = \frac{1}{m_1(n_1+n_2) - m_2 n_1}$.

That is, every solution of (2.1) is also a solution of (2.2) and vice versa.

Proof. By ${}^{C}D^{q}x(t) = y(t)$, $t \in (0, 1)$, and the boundary conditions $x''(0) = x'''(0) = \cdots = x^{(n-1)}(0) = 0$, we have

$$\begin{aligned} x(t) &= I^{q}y(t) + x(0) + x'(0)t + \frac{x''(0)}{2!}t^2 + \cdots + \frac{x^{(n-1)}(0)}{(n-1)!}t^{n-1} \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\mathrm{d}s + x(0) + x'(0)t, \end{aligned} \tag{2.3}$$

where $I^{q}y$ is the fractional integral of order q of the function y.

According to the proposition of the Caputo derivative, we get

$$x'(t) = \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} y(s)\mathrm{d}s + x'(0).$$

Then

$$x(1) = \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} y(s)\mathrm{d}s + x(0) + x'(0),$$

and

$$x'(1) = \frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} y(s)\mathrm{d}s + x'(0).$$