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A new view of the l^p-theory for a system of higher order difference equations

I. Győri *, L. Horváth

Department of Mathematics, University of Pannonia, Egyetem u. 10. 8200 Veszprém, Hungary

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ABSTRACT

It is proved under appropriate assumptions that the solutions of a linear system of higher order difference equations belong to l^p, that is the pth powers of the solutions are summable for some $p \geq 1$. Our results are based on a new transformation of the higher order system into a first-step recursion, where the companion matrices are well treatable from our point of view. Our theory is illustrated by examples, including an interesting exponential stability result and a class of linear delay difference equations.

1. Introduction and notations

In this work, we find new sufficient conditions so that every solution of the d-dimensional system of sth order difference equation

$$x(n) = \sum_{j=n-s}^{n-1} B(n,j)x(j), \quad n \geq 0, \tag{1}$$

belongs to an l^p space, that is $\sum_{n=0}^{\infty} \|x(n)\|^p < \infty$ with a fixed $p \geq 1$.

At first glance, it seems to be an almost trivial observation that system (1) can be reformulated as a ds-dimensional system of first order difference equations in an appropriate sequence space. The matrices of the one-step ds-dimensional system are called companion matrices of Eq. (1). It is clear that they are block matrices defined by all coefficients $B(n,j)$ of Eq. (1). Nevertheless, the obvious advantage of such a reformulation is based on the fact that every solution of the one-step system can be written as a product of the coefficient matrices multiplied by the initial vector of the solution. For the standard reformulation and the technical details we refer to the book by Elaydi [1].

In the present paper, based on a suitable auxiliary vector, a special ds-dimensional one-step difference system is introduced to Eq. (1). Here the companion matrices are much more treatable from our point of view than in the standard transformation case. By using this new ds-dimensional one-step system representation of Eq. (1) and some matrix techniques explicit sufficient conditions are given for the solutions of Eq. (1) to be in l^p.

We note that there are not really many publications on the l^p solutions of s-steps difference equations and the present paper partially fills up this gap. Our paper features an alternative way, inspired by earlier researchers. Gordon [2] obtains criteria for l^p solutions in terms of a Lyapunov function, Petropoulou and Siafarikas [3] consider essentially the space of square summable sequences l^2, and in a recent paper Ey and Pötzsche [4] use fixed point theorems for nonlinear one-step recursions. l^p solutions of nonlinear Volterra difference equations are discussed by Gil' and Medina in [5,6].

* Corresponding author. Tel.: +36 88 624227.
E-mail addresses: gyori@almos.uni-pannon.hu (I. Győri), lhorvath@almos.uni-pannon.hu (L. Horváth).

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Our article is essentially subdivided into five parts.

Section 2 is fundamental for our work and contains basic results on the transformation of Eq. (1) into a one-step system with tractable companion matrices.

The first main result in Section 3 gives a necessary and sufficient condition in terms of our companion matrices for the solutions to be in l^p. The second main result gives explicit conditions in terms of the coefficients $B(n, j)$ of the original Eq. (1), which are therefore easy to check.

In Section 4 some illustrative examples are given to show the effectiveness of our method for the scalar delay difference equation

$$x(n) = \sum_{l=1}^{m} a_l(n)x(n - \sigma_l), \quad n \geq 0,$$

where $a_l(n) \in \mathbb{R}(n \geq 0, l = 1, \dots, m)$ and $1 \leq \sigma_1 < \cdots < \sigma_m$. One of our examples shows that our method is also applicable to get new exponential stability conditions. Our condition is compared to a recent result in the paper Berezansky and Braverman [7].

The proofs of the main results are given in Section 6 based on some preliminary statements (some of them are interesting in their own right) stated and proved in Section 5.

For a positive integer d, $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$ denote the n-dimensional space of column vectors and the d by d matrices with real entries, respectively. Let $\|\cdot\|$ be any norm on $\mathbb{R}^d$. $\mathbb{R}^d$ can be endowed with many norms, but they are all equivalent. The induced norm of a matrix $A \in \mathbb{R}^{d\times d}$ is defined by $\|A\| := \sup\{\|Ax\| \mid x \in \mathbb{R}^d, \|x\| = 1\}$.

Definition 1. (a) The sd-dimensional real vector space of block vectors with entries in $\mathbb{R}^d$ is denoted by V.

(b) The real vector space of $s \times s$ block matrices with entries in $\mathbb{R}^{d\times d}$ is denoted by M.

(c) Let $p \geq 1$. The real Banach space l^p consists of all sequences $u := (u(n))_{n\geq 0}$ in $\mathbb{R}^d$ for which $\sum_{n=0}^{\infty} \|u(n)\|^p < \infty$ with some norm $\|\cdot\|$ on $\mathbb{R}^d$.

(d) Let $p \geq 1$. The real Banach space l_V^p consists of all sequences $v := (v(n))_{n\geq 0}$ in V for which $\sum_{n=0}^{\infty} \|v(n)\|^p < \infty$ with some norm $\|\cdot\|$ on V.

(e) Let $p \geq 1$. The real Banach space l_M^p consists of all sequences $A := (A(n))_{n\geq 0}$ in M for which $\sum_{n=0}^{\infty} \|A(n)\|^p < \infty$ with some norm $\|\cdot\|$ on M.

The elements of $\mathbb{R}^d$ and V are considered as column vectors.

Definition 2. (a) The zero matrix and the identity matrix in $\mathbb{R}^{d\times d}$ are denoted by O and I, respectively.

(b) $\mathcal{O}$ and $\mathcal{I}$ mean the zero matrix and the identity matrix in M, respectively.

If $a \in \mathbb{R}$, then $[a]$ denotes the largest integer that does not exceed a.

2. New phase space representation of the solutions

Consider the system of the sth order difference equations

$$x(n) = \sum_{j=n-s}^{n-1} B(n, j)x(j), \quad n \geq 0 \tag{2}$$

or its equivalent form

$$x(n) = \sum_{j=-s}^{-1} B(n, n + j)x(n + j), \quad n \geq 0, \tag{3}$$

where

(A_1) $s \geq 1$ is a given integer, and $B(n, j) \in \mathbb{R}^{d\times d}(n \geq 0, n - s \leq j \leq n - 1)$.

It is clear that the solutions of (2) are uniquely determined by their initial values

$$x(n) = \varphi(n), \quad -s \leq n \leq -1, \tag{4}$$

where $\varphi(n) \in \mathbb{R}^d(-s \leq n \leq -1)$.

The unique solution of the initial value problem (2) and (4) is denoted by $x(\varphi) = (x(\varphi)(n))_{n\geq -s}$, where $\varphi := (\varphi(-s), \dots, \varphi(-1))^T \in V$.

Let $(x(n))_{n\geq -s}$ be a given sequence in $\mathbb{R}^d$. Then for any fixed $n \geq 0$ we introduce an sd-dimensional state vector $x_n = (x_n(-s), \dots, x_n(-1))^T \in V$ defined by $x_n(i) := x(n + i)(-s \leq i \leq -1)$.

By using the state vector notation, Eq. (2) may be written as an sd-dimensional system of first order difference equations

$$x_{n+1} = \mathcal{B}(n)x_n, \quad n \geq 0$$