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Numerical solution of the higher-order linear Fredholm integro-differential-difference equation with variable coefficients

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ABSTRACT

The main aim of this paper is to apply the Legendre polynomials for the solution of the linear Fredholm integro-differential-difference equation of high order. This equation is usually difficult to solve analytically. Our approach consists of reducing the problem to a set of linear equations by expanding the approximate solution in terms of shifted Legendre polynomials with unknown coefficients. The operational matrices of delay and derivative together with the tau method are then utilized to evaluate the unknown coefficients of shifted Legendre polynomials. Illustrative examples are included to demonstrate the validity and applicability of the presented technique and a comparison is made with existing results.

1. Introduction

Integro-differential equations have gained a lot of interest in many applications, such as biological, physical, and engineering problems. The numerical methods for solution of Fredholm integro-differential equations have been investigated in many studies [1–4].

In recent years, a lot of attention has been devoted to the study of differential-difference equations, i.e., equations containing shifts of the unknown function and its derivatives, and also integro-differential-difference equations. For instance, see [5–7]. These equations occur frequently as a model in mathematical biology and the physical sciences [8,9]. Also partial integro-differential equation is a good model for viscoelasticity (see for example [10] and the references therein). The interested reader can see [4,11–17] for more research works on the numerical solution of integral equations.

In this work, we develop a framework to obtain the numerical solution of the sth-order linear Fredholm integro-differential-difference equation with variable coefficients

$$\sum_{k=0}^{s} p_k(x)y^{(k)}(x) + \sum_{r=0}^{t} p_r^*(x)y^{(r)}(x-\tau) = f(x) + \int_a^b K(x,t)y(t-\tau)\mathrm{d}t, \quad \tau \geq 0, \tag{1}$$

with the mixed conditions

$$\sum_{k=0}^{s-1}[\alpha_{ik}y^{(k)}(a) + \beta_{ik}y^{(k)}(b) + \gamma_{ik}y^{(k)}(\eta)] = \mu_i, \quad i = 0, 1, \ldots, s-1, \tag{2}$$

where $p_k(x)$, $p_r^*(x)$, $K(x,t)$ and $f(x)$ are known continuous functions. Here the real coefficients α_{ik}, β_{ik}, γ_{ik} and μ_i are appropriate constants. Note that η is a given point in the spatial domain of the problem. This problem has been recently

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considered by [9]. As described in [9], high-order linear Fredholm integro-differential-difference equations with variable coefficients are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. In [9] the authors approximate the unknown function $y(x)$ by Taylor series and transform the equation and the given conditions into the matrix equations. By solving a system of linear algebraic equations, the Taylor coefficients of the solution function is obtained. Also a Taylor method has been extended to solve the Fredholm integro-differential equations [18]. The interested reader can see [19,20] for more published research works in the subject.

Our approach consists of reducing the problem to a set of linear equations by expanding the approximate solution $y(x)$ in terms of Legendre polynomials with unknown coefficients. The operational matrices of delay and derivative are given. These matrices together with the tau method are then utilized to evaluate the unknown coefficients of Legendre polynomials. The tau method has been originally proposed by Lanczos [21] for ordinary differential equations and extended by Ortiz [22]. The method consists of expanding the required approximate solution as the elements of a complete set of orthogonal polynomials [23,24]. Recently there have been several published works in the literature on the applications of the tau method [25–28]. For more details of Legendre polynomials see [29,30] and also some technique for solving integro-differential equations can be found in [31,32].

The organization of the rest of this paper is as follows: Section 2 is devoted to the basic formulation of Legendre polynomials required for our subsequent development. Section 3 summarizes the application of Legendre tau method to the solution of problem (1)–(2). Thus, a set of linear equations is formed and a solution of the considered problem is introduced. In Section 4 the proposed method is applied to several numerical examples and a comparison is made with existing methods in the literature. Section 5 concludes the paper. Note that we have computed the numerical results by Maple programming.

2. Properties of shifted Legendre polynomials

The well-known Legendre polynomials are defined on the interval $z \in [-1, 1]$ and can be determined with the aid of the following recurrence formulae:

$$L_{i+1}(z) = \frac{2i+1}{i+1} z L_i(z) - \frac{i}{i+1} L_{i-1}(z), \quad i = 1, 2, \ldots$$

with $L_0(z) = 1$ and $L_1(z) = z$. For practical use of Legendre polynomials on the interval of interest $x \in [a, b]$, it is necessary to shift the defining domain by means of the following substitution:

$$z = \frac{2x - a - b}{b - a}, \quad a \le x \le b.$$

The shifted Legendre polynomials in x are then obtained as follows:

$$L_0(x) = 1, \qquad L_1(x) = \frac{2x - a - b}{b - a},$$

$$L_{i+1}(x) = \frac{(2i+1)(2x - a - b)}{(i+1)(b-a)} L_i(x) - \frac{i}{i+1} L_{i-1}(x), \quad i = 1, 2, \ldots. \tag{3}$$

The orthogonality condition is

$$\int_a^b L_i(x) L_j(x) \mathrm{d}x = \begin{cases} \dfrac{b-a}{2i+1}, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \tag{4}$$

A function $y(x)$, square integrable in $[a, b]$, may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{j=0}^{\infty} c_j L_j(x),$$

where the coefficients c_j are given by

$$c_j = \frac{2j+1}{b-a} \int_a^b y(x) L_j(x) \mathrm{d}x, \quad j = 1, 2, \ldots.$$

In practice, only the first $(m + 1)$-terms shifted Legendre polynomials are considered. Then we have

$$y_m(x) = \sum_{j=0}^{m} c_j L_j(x) = \Phi^T(x) C,$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\Phi(x)$ are given by

$$C = [c_0, c_1, \ldots, c_m]^T,$$

$$\Phi(x) = [L_0(x), L_1(x), \ldots, L_m(x)]^T. \tag{5}$$