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Nonlinear multigrid methods for second order differential operators with nonlinear diffusion coefficient

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ABSTRACT

Nonlinear multigrid methods such as the Full Approximation Scheme (FAS) and Newton-multigrid (Newton-MG) are well established as fast solvers for nonlinear PDEs of elliptic and parabolic type. In this paper we consider Newton-MG and FAS iterations applied to second order differential operators with nonlinear diffusion coefficient. Under mild assumptions arising in practical applications, an approximation (shown to be sharp) of the execution time of the algorithms is derived, which demonstrates that Newton-MG can be expected to be a faster iteration than a standard FAS iteration for a finite element discretisation. Results are provided for elliptic and parabolic problems, demonstrating a faster execution time as well as greater stability of the Newton-MG iteration. Results are explained using current theory for the convergence of multigrid methods, giving a qualitative insight into how the nonlinear multigrid methods can be expected to perform in practice.

1. Introduction

Nonlinear multigrid iterations such as the Full Approximation Scheme (FAS) [1] and Newton-Multigrid (Newton-MG) [2,3] methods have been widely used to solve elliptic and parabolic nonlinear problems on large scales (cf. [4–9] amongst others). There exists very little convergence theory for the nonlinear methods (e.g. [10–13]) and only limited comparison as to which method should be preferred in practice [14–18], where the comparisons are limited to specific applications. In this paper we present a more general framework for comparing the relative efficiency of the Newton-MG and FAS methods for a broad class of nonlinear problems. This requires a detailed discussion of the efficient implementation of these schemes, followed by a theoretical assessment of their running times in a finite element setting for a general second order nonlinear operator. The comparison is based upon a detailed analysis of their costs per cycle, followed by a theoretical discussion of their convergence properties and how this theory may be used when comparing the techniques. As there exists no algebraic variant of FAS multigrid, the geometric algorithms are compared.

The remainder of the paper is structured as follows. In Section 2 we briefly present the linear and nonlinear multigrid iterations, followed by a detailed discussion of the theoretical running time of Newton-MG and FAS in Section 3. Section 4 describes and applies the relevant convergence theory for linear and nonlinear multigrid iterations. In Section 5 model problems are introduced, which are used to produce results in Section 6, to demonstrate the applicability of the theory from Sections 3 and 4. Conclusions are given in Section 7, which summarise the reasons why a Newton-MG iteration should be preferred over an FAS iteration when using a finite element discretisation.

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2. Background

In this section we introduce the basic concepts and notation required for the definition of both linear and nonlinear geometric multigrid algorithms. A more detailed introduction can be found in [2,11,3]. The problem to be solved is presented as an operator equation. Once an operator is discretised, and an appropriate basis for a discrete subspace has been chosen, the discrete operator equation may be considered an algebraic system of equations. In the following we move between considering operator equations and the corresponding algebraic systems of equations, as appropriate.

2.1. Linear multigrid algorithms

We wish to solve the linear operator equation given by

$$\mathcal{A}u(x) = f(x), \quad x \in \Omega \tag{2.1}$$

where the domain $\Omega \in \mathbb{R}^d$ has boundary $\partial\Omega$, and $\mathcal{A} : \mathcal{V} \to \mathcal{V}$ for some vector space $\mathcal{V}$. From this point on we omit the explicit dependence on $x \in \mathbb{R}^d$. It is assumed that there is a unique $u^* \in \mathcal{V}$ satisfying Eq. (2.1). We are interested in the approximate solution of (2.1) based upon discretisations using a sequence of finite-dimensional grids

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_J, \tag{2.2}$$

which are sets of connected points in Ω. We also consider the sequence of finite-dimensional function spaces

$$V_1 \subset V_2 \subset \cdots \subset V_J \subset \mathcal{V}, \tag{2.3}$$

where each $V_l,\ l = 1, \ldots, J$ is defined on grid Ω_l. Given (2.3), we consider the discretised system of equations

$$A_l u_l = f_l \tag{2.4}$$

where $A_l : V_l \to V_l$ is the projection of the continuous operator $\mathcal{A}$ onto the finite-dimensional space V_l. We assume there are unique $u_l^* \in V_l,\ l = 1, \ldots, J$ that satisfy (2.4). For the purposes of this paper $V_l,\ l = 1, \ldots, J$ is the standard piecewise linear finite element function space defined on grid Ω_l.

2.1.1. Linear multigrid as a solver

In this section we describe the linear geometric multigrid method and introduce some notation. A discussion of necessary conditions for convergence of geometric multigrid is presented in Section 4.

We introduce operators

$$\begin{aligned} &R_l : V_l \to V_{l-1}, \quad l = 2, \ldots, J \\ &P_l : V_{l-1} \to V_l, \quad l = 2, \ldots, J, \end{aligned} \tag{2.5}$$

which are restriction and prolongation operators, respectively, that allow the transfer of functions between different subspaces. Since the exact solution to (2.4) is $u_l^* \in V_l$, the error e_l and defect r_l in approximation u_l, defined by

$$e_l = u_l^* - u_l, \qquad r_l = f_l - A_l u_l,$$

satisfy the operator equation

$$A_l e_l = r_l. \tag{2.6}$$

We assume there exist operators $S_l : V_l \to V_l,\ l = 2, \ldots, J$, called smoothing operators, that have the property that they are effective at removing high frequency components from the error [3,2]. A correction term is calculated on a coarser grid in the *coarse grid correction* step. We fix a number ν of smooths to perform before (pre-smoothing) and after (post-smoothing) a coarse grid correction step. In general the number of pre- and post-smooths may differ, and one of the smoothing steps may be left out entirely [11,3,2].

Consider that we wish to solve (2.4) on grid $\Omega_l,\ l \neq 1$, for the exact discrete solution $u_l^* \in V_l$. A single step of the geometric multigrid algorithm is then outlined in Algorithm 2.1, where $u_l^{(j)} \in V_l$ represents the approximation to the solution u_l^* after j iterations of linear multigrid have been performed. This iteration can be performed until some appropriate convergence/failure criteria are met. On the coarsest grid level the exact inverse is very inexpensive to compute, provided that $|\Omega_1|$ is small, and the running time of the algorithm is $O(n)$ for particular choices of A (cf. [3,2]). There is a closed form representation of the linear multigrid V-cycle operator, which for the rest of this paper is denoted $M_l : V_l \to V_l$. The exact representation can be found in [3]. A multigrid operator should possess the smoothing and coarse grid correction properties [11]. That is, the smoother should remove high frequency components from the error, and the coarse grid correction should give a good approximation to the fine grid error after the high frequency components have been removed.

2.1.2. Multigrid preconditioned linear iterations

Multigrid iterations are frequently used as a preconditioner for a different iterative method. For a more thorough discussion of such methods see [19]. The discrete problem (2.4) can be solved iteratively using a right-preconditioned Krylov subspace method, such as conjugate gradients (CG) or GMRES [19]. As the preconditioner we use a single multigrid V-cycle