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First order least squares method with weakly imposed boundary condition for convection dominated diffusion problems

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ABSTRACT

We present and analyze a first order least squares method for convection dominated diffusion problems, which provides robust L^2 a priori error estimate for the scalar variable even if the given data $f \in L^2(\Omega)$. The novel theoretical approach is to rewrite the method in the framework of discontinuous Petrov–Galerkin (DPG) method, and then show numerical stability by using a key equation discovered by Gopalakrishnan and Qiu (2014). This new approach gives an alternative way to do numerical analysis for least squares methods for a large class of differential equations. We also show that the condition number of the global matrix is independent of the diffusion coefficient. A key feature of the method is that there is no stabilization parameter chosen empirically. In addition, Dirichlet boundary condition is weakly imposed. Numerical experiments verify our theoretical results and, in particular, show our way of weakly imposing Dirichlet boundary condition is essential to the design of least squares methods—numerical solutions on subdomains away from interior layers or boundary layers have remarkable accuracy even on coarse meshes, which are unstructured quasi-uniform.

1. Introduction

In this paper, we present a robust a priori analysis of first order least squares method with weakly imposed boundary condition for the following convection dominated diffusion equation

$$-\epsilon \Delta u + \boldsymbol{\beta} \cdot \nabla u + cu = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\Omega \in R^d$ $(d = 2, 3)$ is a polyhedral domain, $0 < \epsilon \le 1$, c a function in $L^\infty(\Omega)$, f a function in $L^2(\Omega)$ and g a function in $H^{1/2}(\partial\Omega)$. Here, the variable flux $\boldsymbol{\beta}$ satisfies the following assumption:

$$\boldsymbol{\beta} \cdot \nabla \psi \ge b_0 > 0 \text{ in } \Omega, \text{ for some function } \psi \in W^{1,\infty}(\Omega), \tag{1.2a}$$

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$$c - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \geq 0 \text{ in } \Omega. \tag{1.2b}$$

According to [1], the assumption (1.2a) is satisfied if

$$\boldsymbol{\beta} \text{ has no closed curves and } |\boldsymbol{\beta}(x)| \neq 0 \text{ for all } x \in \Omega.$$

Least squares methods have been frequently used to simulate solutions of partial differential equations arising from fluid dynamics and continuum mechanics. We refer to [2,3] for comprehensive summary. It is well known that least squares methods have the following desirable features: it leads to a minimization problem; its numerical stability is *not* sensitive to the choice of finite element space or meshes; the resulting global stiffness matrices are symmetric and positive definite; a practical a posteriori error estimator can be given without any additional cost, and so on (see [4–16]).

Unfortunately, primitive least squares methods for convection dominated diffusion problems (1.1) have the following drawbacks. Firstly, if the term $c - \frac{1}{2}\nabla \cdot \boldsymbol{\beta}$ is *not* uniformly bounded from below by a positive constant, L^2 a priori error estimate of primitive least squares methods will deteriorate as the diffusion coefficient ϵ goes to zero, even when the exact solution has no interior layers or boundary layers (see error estimates in [17–19]). Secondly, primitive least squares methods show a very poor performance for convection diffusion problem (1.1) with a sufficiently small diffusion coefficient, because large spurious oscillations are observed (see numerical experiments in [18]). We notice that in [18], residual-free bubble strategy is used to address the second drawback. But, the least squares method in [18] needs to compute basis functions element-wise, which is relatively not easy to implement.

It is well known that streamline diffusion method [20], residual free bubble methods [21–23], and DG methods [1,24–27] do not suffer from the above two drawbacks of primitive least squares methods. We refer to [28,29] as comprehensive summaries of numerical methods suitable for convection dominated diffusion problems. We would like to emphasize that none of these numerical methods (streamline diffusion method, residual free bubble methods, DG methods) result in symmetric global stiffness matrices. Hence from the point of view of solver design, the least squares method is more attractive than the other methods mentioned before and many works have been contributed to this subject (e.g. [9,30]). Moreover, we derive that the condition number of linear system from our first order least squares method is at most $\mathcal{O}(h^{-2})$, where h is the mesh size. In particular, the condition number is independent of the diffusion coefficient. This property is important for designing efficient solver, e.g., multilevel method, for the first order least squares approximation of convection dominated diffusion equation.

In this paper, we propose and analyze our first order least squares method to address these two drawbacks for primitive least squares methods. In fact, it is difficult to provide robust L^2 error estimate by the traditional approach of numerical analysis for least squares methods in [2]. So, it is necessary to look for an alternative approach. We notice that a weighted test function was used in [31] to obtain the L^2 stability of the original DG method [32] for the transportation reaction equation, and this idea was generalized to convection–diffusion–reaction equation in [1] using the IP–DG methods. In this paper, we rewrite our method in the framework of discontinuous Petrov–Galerkin (DPG) method, then show numerical stability by using a key equation discovered in [33]. The advantage of this new approach is that the weight function in [1] is shown to stay in some “equivalent” test function space (see (3.7) in Section 3) such that numerical stability can be obtained without using any projection as in [1]. This approach is novel and useful to numerical analysis of least squares methods for a large class of differential equations. This new approach of numerical analysis is also different from traditional ones used for DPG method in [34–38]. We show that, roughly speaking, using polynomials of degree $k+1 \geq 1$,

$$\|u_h - u\|_{L^2(\Omega)} + \epsilon^{1/2}\|\nabla(u-u_h)\|_{L^2(\Omega)} \leq Ch^{k+1}\|u\|_{H^{k+2}(\Omega)}; \tag{1.3}$$

if $\epsilon^{1/2} \leq h_K$ for any $K \in \mathcal{T}_h$,

$$\|u - u_h\|_{L^2(\Omega)} + \epsilon^{1/2}\|\nabla(u-u_h)\|_{L^2(\Omega)} + \|\boldsymbol{\beta}\cdot\nabla(u-u_h)\|_{L^2(\Omega)} \leq Ch^{k+1}\|u\|_{H^{k+2}(\Omega)}.$$

Here, the constant C is independent of ϵ. Thus, we can conclude that a priori error estimate in (1.3) is *robust* with respect to the diffusion coefficient ϵ, which addresses the first draw back. We also want to emphasize that the convergence result (1.3) shows our method has L^2 convergence rate even if $f \in L^2(\Omega)$, which means our method does not have excessive smoothness requirements than other methods. In order to overcome the second drawback, we impose Dirichlet boundary condition in a weak way, such that the error along the boundary layers will not propagate into the whole domain. We show the advantage of imposing boundary condition weakly by numerical experiments. We notice that our way of imposing boundary condition is similar to the weak imposition of Dirichlet boundary condition in [39–42] (in [41], boundary condition is weakly imposed for least squares methods for linear hyperbolic PDEs), which belongs to Nitsche's method in [43]. However, we do *not* have to choose any penalty terms empirically while [40] needs. We would like to emphasize that weakly imposing boundary condition is essential to least squares methods while it is incrementally helpful to streamline diffusion method and DG methods (see numerical experiments in [40]). If boundary condition is imposed strongly, the numerical solutions produced by streamline diffusion method and DG methods may have artificial oscillation along boundary layers, while the accuracy in subdomains away from boundary layers is still remarkable. However, according to our numerical experiments, if we impose boundary condition strongly, then numerical solution of least squares methods will be polluted on almost the whole domain by boundary layers. We have tried to add several stabilization terms, which have been utilized by streamline diffusion