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Computation of Yvon-Villarceau circles on Dupin cyclides and construction of circular edge right triangles on tori and Dupin cyclides

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ABSTRACT

Ring Dupin cyclides are non-spherical algebraic surfaces of degree four that can be defined as the image by inversion of a ring torus. They are interesting in geometric modeling because: (1) they have several families of circles embedded on them: parallel, meridian, and Yvon-Villarceau circles, and (2) they are characterized by one parametric equation and two equivalent implicit ones, allowing for better flexibility and easiness of use by adopting one representation or the other, according to the best suitability for a particular application. These facts motivate the construction of circular edge triangles lying on Dupin cyclides and exhibiting the aforementioned properties. Our first contribution consists in an analytic method for the computation of Yvon-Villarceau circles on a given ring Dupin cyclide, by computing an adequate Dupin cyclide-torus inversion and applying it to the torus-based equations of Yvon-Villarceau circles. Our second contribution is an algorithm which, starting from three arbitrary 3D points, constructs a triangle on a ring torus such that each of its edges belongs to one of the three families of circles on a ring torus: meridian, parallel, and Yvon-Villarceau circles. Since the same task of constructing right triangles is far from being easy to accomplish when directly dealing with cyclides, our third contribution is an indirect algorithm which proceeds in two steps and relies on the previous one. As the image of a circle by a carefully chosen inversion is a circle, and by constructing different images of a right triangle on a ring torus, the indirect algorithm constructs a one-parameter family of 3D circular edge triangles lying on Dupin cyclides.

1. Introduction

3D (triangular) meshes are piece-wise linear representations of the boundaries of sets in the affine Euclidean space $\mathbb{E}^3$. They constitute a very popular representation in computer graphics and geometry fields because of their compactness compared to volumetric representations (e.g., voxel grids), the ease of their rendering which can be hardware accelerated (exclusive use of triangles), their approximation power, and their simplicity compared to other higher order boundary representations such as B-splines and Bézier surfaces. However, meshes present some disadvantages, such as the high number of triangles required for a faithful representation of the fine details of objects and the inherent heavy edition and visualization. For instance, moving a mesh vertex requires an unobvious and challenging update of the neighboring vertices coordinates, in addition to paying attention to topological issues like interpenetration of moving mesh boundaries.

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Visualization is also problematic when the viewer moves towards the mesh since more details are required but lacking in the mesh itself. On the other hand, a highly detailed mesh is not necessary for a viewer moving away from it and these details tend to slow down scene rendering and visualization. These issues have motivated the development of algorithms allowing the passage between different levels of details (simplification and subdivision) for meshes, according to the prescribed use. These algorithms are costly and exhibit principal drawbacks consisting in either the introduction of information that was not part of the (approximated) object in the case of subdivision, or the removal of information from the object (simplification), which is even worse.

In order to address the aforementioned issues, it is interesting to consider a higher level approximation of objects, by appropriately grouping mesh triangles into several high degree surface patches exhibiting specific characteristics. Doing so reduces the number of triangles needed to achieve the same level of detail (fewer patches). However, to ease edition and visualization, one must keep the degree of surface patches relatively low (four or less) and to ideally adopt surfaces having both parametric and algebraic representations, in order to use either one representation or the other, depending on which one fits best the task at hand. It turns out than Dupin cyclides and triangles embedded on them have all these desirable characteristics and thus are good candidates to address mesh-related problems.

Dupin cyclides are non-spherical algebraic surfaces of degree four introduced in 1822 by Charles Dupin [1]. They can be defined as the envelope of two families of one parameter oriented spheres. In consequence, all their curvature lines are circular because they are generated by spheres belonging to the aforementioned families [2,3]. According to [4,3], Dupin cyclides are characterized by one easy to understand parametric equation (only 3 parameters) and by two equivalent implicit ones. These properties make them very interesting in geometric modeling. In 1982, Ralph Martin was the first person who introduced Dupin cyclides in geometric design [5] when he used them for the formulation of his principal patches. Later on, Dupin cyclides attracted a lot of attention and their algebraic and geometric properties have been extensively investigated [6–10,4].

The fact that Dupin cyclides possess both an easy to understand parametric representation and two low-degree equivalent implicit representations is advantageous because it allows for more flexibility and easiness of use, depending on the application domain. On the one hand, the low-degree implicit representations are preferable for many geometric algorithms and allow the development of robust and efficient solutions to a wide range of tasks, such as derivatives and tangents computation, ray-shooting, point-on-surface queries, intersections, and lines of curvature determination. On the other hand, the parametric representation fits better the editing, animation, and visualization of surfaces. Another important argument encouraging Dupin cyclides use is their well-established conversions to other parametric surfaces, such as Bézier curves, B-splines, and NURBS [4,11–15]. This point bridges the gap between the different types of surfaces and may accelerate the introduction of Dupin cyclides into modeling systems based on parametric surfaces.

From a modeling point of view, it is clear that a triangle embedded on a Dupin cyclide (called cycloidal triangle in the sequel) approximates the surface of a 3D object better than a classical triangle (called planar triangle). Added to that, constructing an object by assembling cycloidal triangles requires less topological considerations compared to meshes, because a much more smaller number of cycloidal triangles is necessary, implying a small number of joints to be handled.

When coming to visualization, it appears that rendering cycloidal triangles is more accurate than rendering meshes composed of planar triangles, whose surface is only G^0 continuous (non-smooth visualization). Moreover, Ray-tracing a cycloidal triangle is much more simpler than computing the intersections of a ray and a set of planar triangles, even if the former operation implies solving a quartic equation. This is justified by the relatively high number of planar triangles required for achieving a similar quality of visualization.

In this work, we present three contributions related to Yvon-Villarceau circles computation and circular edge right triangles construction on tori and Dupin cyclides. First, thanks to an adequate Dupin cyclide-torus inversion, we present an analytic method for the computation of implicit and parametric equations of Yvon-Villarceau circles on ring Dupin cyclides. Yvon-Villarceau circles embed one of the three circular edges of cycloidal triangles and their computation is involved in our third contribution. Second and third, given three points in $\mathbb{E}^3$, we propose two algorithms for the construction of circular edge right triangles on ring tori (called toroidal triangles in the sequel) and Dupin cyclides (cycloidal triangles), whose vertices correspond to the given points. The first algorithm solves a quartic equation in order to compute toroidal triangles, while the second relies on the first one, computes an intermediate toroidal triangle, and computes the image by inversion of the later in order to obtain cycloidal triangles, ensuring the invariance of the input points. In this work, we exclusively deal with right triangles. However, in order to simplify our discussion, we sometimes omit the qualifier "right" when referring to toroidal and cycloidal triangles.

To the best of our knowledge, the computation of Yvon-Villarceau circles on ring Dupin cyclides and the construction of toroidal triangles have not yet been accomplished previously. Regarding cycloidal triangles construction, only one related work has been proposed in literature [16]. In this work, Belbis et al. presented a three-steps algorithm for the construction of circular edge triangles on Dupin cyclide patches. Given four input points, they first constructed a biquadratic rational Bézier surface representing a Dupin cyclide patch, whose vertices correspond to the input points. Then, they determined the parameters of the cyclide and computed meridian and parallel circles as the edges of the constructed patch. Finally, if there exists an Yvon-Villarceau circle passing through two diagonal vertices of the patch, then two cycloidal triangles are constructed. Otherwise, Belbis et al. computed the point intersection of a Yvon-Villarceau circle passing through a vertex of the patch with either a meridian or a parallel circle and considered it as the third vertex of the cycloidal triangle. The main drawback of the approach presented in [16] is that only two vertices can be constrained (i.e., can be guaranteed to coincide