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A blow-up result for nonlinear generalized heat equation

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ABSTRACT

In this paper we consider a nonlinear heat equation with nonlinearities of variable-exponent type. We show that any solution with nontrivial initial datum blows up in finite time. We also give a two-dimension numerical example to illustrate our result.

1. Introduction

In this paper we are interested in the existence and the blow-up in finite time of solutions of the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) = |u|^{p(x)-2}u + f & \text{in } Q = \Omega \times (0,T), \\ u = 0 & \text{on } \partial Q = \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $m(.)$ and $p(.)$ are two continuous functions on $\overline{\Omega}$ such that:

$$2 \le m_- \le m(x) \le m_+ < p_- \le p(x) \le p_+ < m_*(x), \tag{1.1}$$

with $m_*(x) = \begin{cases} \dfrac{Nm(x)}{(N-m(x))_+} & \text{if } m_+ < N, \\ +\infty & \text{if } m_+ \ge N. \end{cases}$

We also assume that

$$|m(x) - m(y)| \le \frac{A}{\log\left(\frac{1}{|x-y|}\right)}, \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta \tag{1.2}$$

with $A > 0,\ 0 < \delta < 1$ and

$$\operatorname*{ess\,inf}_{x\in\Omega}(m^*(x) - p(x)) > 0. \tag{1.3}$$

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When $m(x) = m$, $p(x) = p$, existence and blow-up results have been established by many authors (see [1–6]). For instance, Fujita in [4] considered the case where

$$\partial_t u = \Delta u + u^p, \quad \text{in } \mathbf{R}^N$$

and proved the existence of a critical exponent $p_c = 1 + \frac{2}{N}$ such that:

If $1 < p < p_c$, then the only nonnegative global (in time) solution is $u = 0$.
If $p > p_c$, then there exists a global solution for sufficiently small initial values.

Levine et al. [7] considered the generalized mean curvature, when $\Omega = \mathbf{R}^N$ and when the elliptic operator is of the form $\text{div}(\psi(1+|\nabla u|^2)^{1/2}\nabla u)$ with ψ is a real function satisfying some specific conditions, and established the following results:

If $1 < p < p_c$, then there are no nontrivial positive solutions.
If $p > p_c$, then there exist both a positive global solution, and solutions which blow up in finite time; however no conclusion was obtained when $p = p_c$.

Ôtani [6] studied the existence and the asymptotic behavior of solutions of (P) and overcome the difficulties caused by the use of nonmonotone perturbation theory. The quasilinear case, with $m \neq 2$, requires a strong restriction on the growth of the forcing term $|u|^{p-2}u$, which is caused by the loss of the elliptic estimate for the m-Laplacian operator defined by $\Delta_m u = \text{div}(|\nabla u|^{m-2}\nabla u)$ (see [2]). Recently in [1], Agaki proved an existence and blow up result for the initial datum $u_0 \in L^r(\Omega)$.

In the case where $m(.)$ and $p(.)$ are two measurable functions, some different techniques must be adopted to study the existence and blow-up of the problem (P) and the traditional methods may fail unless some modifications are made.

Let us mention that the study of differential equations with nonstandard $p(x)$-growth is a new and interesting topic. It arises from the nonlinear elasticity theory, electrorheological fluids, etc. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. For a general account of the underlying physics, we refer the reader to [8] and for the mathematical theory see [9]. In this context, a series of papers by Diening and Ružicka [10,11] related to problems in the so-called rheological and electrorheological fluids, which lead to spaces with variable exponents, have appeared lately. The results developed in those papers were collected in the books [12,13]. Many mathematical models in fluid mechanics, elasticity theory (recently in image processing, see for example [14]), etc. were shown to be naturally related to the problems with non-standard local growth.

Our objective in this paper is to study the blow up phenomenon of solutions of the problem (P) in the framework of the Lebesgue and Sobolev spaces with variable exponents. We will establish a blow up result and give a numerical example in 2D to illustrate our theoretical result.

2. Preliminaries

2.1. Functional framework

We list some well-known results about the Lebesgue and Sobolev spaces with variable exponents (see [13]).
Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of $\mathbf{R}^N$ with $N \geq 2$. We denote by

$$p_- = \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p_+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

The $p(.)$ modular of a measurable function $u : \Omega \to \mathbf{R}^N$ is defined as

$$\varrho_{p(.)} = \int_{\Omega-\Omega_\infty} |u(x)|^{p(x)}\,dx + \operatorname*{ess\,sup}_{x\in\Omega_\infty} |u(x)|,$$

where

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

The variable-exponent Lebesgue space $L^{p(.)}(\Omega)$ consists of all measurable functions u defined on Ω for which

$$\varrho_{p(.)}(\lambda u) < \infty,$$

for some $\lambda > 0$.
The Luxembourg norm on this space is defined as

$$\|u\|_{p(.)} = \inf\{\lambda > 0 : \varrho_{p(.)}(u/\lambda) \leq 1\}.$$

Equipped with this norm, $L^{p(.)}(\Omega)$ is a Banach space (see [13,15]).

Remark 2.1. The variable exponent-Lebesgue space is a special case of more general Orlicz–Musielak spaces. For the constant function $p(x) = p$, the variable exponent Lebesgue space coincides with the classical Lebesgue space.