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Exponential growth of solution of a strongly nonlinear generalized Boussinesq equation

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ABSTRACT

An initial–boundary value problem for strongly nonlinear generalized Boussinesq equation is studied. We show the exponential growth of solution with L_p-norm for negative or positive initial energy by constructing differential inequalities.

1. Introduction

In this paper, we study the following initial–boundary value problem of a class of reaction–diffusion equations with multiple nonlinearities

$$u_t - \Delta u_t - \Delta u + |u|^{k-2}u_t = |u|^{p-2}u, \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $k > 2, p > 2$ are real numbers and Ω is bounded domain in R^n with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied. Here, Δ denotes the Laplace operator in Ω.

This problem was derived in [1] and was called an initial–boundary value problem for the generalized Boussinesq equation. It describes an electric breakdown in crystalline semiconductors with allowance for the linear dissipation of bound- and free-charge sources [2,3].

In the absence of the nonlinear diffusion term $|u|^{k-2}u_t$, Eq. (1.1) reduced to the following equation

$$u_t - \Delta u_t - \Delta u = f(u), \tag{1.4}$$

it was introduced as a general class of equations of Sobolev type, sometimes referred to as Sobolev–Galpern type, it is also referred in the literature as the Benjamin–Bona–Mahony–Burgers' (BBM–Burgers) equation, or a viscous regularization of the original BBM model for long wave propagation, see [4,5]. Eq. (1.4) appears as a nonclassical diffusion equation in fluid mechanics, solid mechanics and heat conduction theory, see for instance [6] and references therein. Since Eq. (1.4) contains

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the term $-\Delta u_t$, it is different from the usual reaction–diffusion equation essentially [6]. Equations of the form (1.4) have been called also pseudo-parabolic because well posed initial–boundary value problems for parabolic equations are also well-posed for (1.4), see [5].

Related problems to Eq. (1.4) have attracted a great deal of attention in the last two decades, and many results have been appeared on the existence, blowup and asymptotic behavior of solutions. For example, Sun [6], Karch [7] and Shishmarev [8] considered the dynamical behavior of Cauchy problem for Eq. (1.4) with critical nonlinearity for both autonomous and nonautonomous cases. Xu and Su [9] proved the invariance of some sets, global existence, nonexistence and asymptotic behavior of solutions for the initial–boundary value problem of (1.4) with $f(u) = u^p$ by introducing a family of potential wells. Korpusov [10–12] gave the sufficient condition of blow-up of solutions of equations (1.4). Especially, Kozhanov [13,14] proved the blow-up using the comparison principle for solutions of the first boundary-value problem for the following Boussinesq equation

$$u_t - \Delta u_t - \Delta g(u) = f(u).$$

The obtained results show that global existence and nonexistence depend roughly on m, the degree of nonlinearity in f, the dimension n, and the size of the initial data. The global existence and nonexistence of solution for (1.4) without the term $-\Delta u_t$ have been investigated by many authors. See in this regard, the works of Levine [15,16], Kalantarov and Ladyzhenskaya [17], Messaoudi [18], Liu et al. [19], Levine et al. [20] and references therein.

Eq. (1.1) without term $-\Delta u_t$ can also be as a special case of doubly nonlinear parabolic-type equations (or the porous medium equation) [21,22,20],

$$\beta(u)_t - \Delta u = |u|^{p-2}u \tag{1.5}$$

if we take $\beta(u) = u + |u|^{m-2}u$. The author of [21,22] took Eq. (1.5) as dynamical systems and studied their attractors. Levine and Sacks [23,24] and Blanchard and Francfort [25] proved the existence and nonexistence for solutions. Zhang [26] and Ding and Guo [27–29] concerned with the following nonlinear parabolic equation with a gradient term and Neumann (or Robin) boundary condition

$$(b(u))_t = \nabla.(a(u) \nabla u) + f(x, u, |\nabla u|^2, t). \tag{1.6}$$

They presented an upper bound of blowup-time by the upper and lower solution methods.

We also point that the authors of [30] used comparison principle and variational methods to study the following equation and obtained the finite time blow-up solution.

$$u_t - \Delta u = |u|^{p-2}u \quad \text{in } \Omega \times (0, T).$$

In [9], the authors studied the IBVP of reaction–diffusion equations with several nonlinear source terms of different signs and got a finite time blow-up of solutions at high energy level.

$$u_t - \Delta u = f(u) = \sum_{k=1}^{l} a_k |u|^{p_k - 1}u - \sum_{j=1}^{s} b_j |u|^{q_j - 1}u, \quad x \in \Omega\ t > 0.$$

In [31], the authors discussed the following equations finite time blow-up solution at high level.

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) + u^{1+\alpha}.$$

Polat [32] established a blow up result for the solution with vanishing initial energy of the following initial–boundary value problem

$$u_t - u_{xx} + |u|^{k-2}u_t = |u|^{p-2}u.$$

They also gave detailed results of the necessary and sufficient blow up conditions together with blow up rate estimates for the positive solution of the problem

$$(u^m)_t - \Delta u = f(u)$$

subject to various boundary conditions. Korpusov and Sveshnikov [2,3,33,34] studied the following problem

$$u_t - \Delta u_t - \Delta u + |u|^{k-2}u_t = f(u),$$

where $f(u) = u(u+\alpha)(u-\beta)$ or $|u|^p u$ with initial–boundary value (1.2) and (1.3) or nonlinear Neumann boundary condition. They gave the local strong solution and the sufficient close-to-necessary conditions for the blowup of solutions to the above problem for negative initial energy using the energy approach developed by Levine [16]. Furthermore, they considered also two different abstract Cauchy problems for equations of Sobolev type.

In this paper, we will investigate the exponential growth of the problem (1.1)–(1.3) with negative or positive initial energy, that is to say we will prove that solutions grow exponentially i.e.

$$\|u_t\|^2 + \|\nabla u\|^2 \to \infty, \quad t \to +\infty.$$